May 23, 2021

Algebra.

Recap: Elements of Set Theory.

Arrangements and Derangements.

Arrangements of a subset of k distinct objects chosen from a set of n distinct objects are $A_n^k = \frac{n!}{(n-k)!}$ permutations [order matters] of distinct subsets of k elements chosen from that set. The total **number of arrangements** of all subsets of a set of n distinct objects is the number of unique sequences [order matters] that can be formed from any subset of $0 \leq k \leq n$ objects of the set,

$$a_n = \sum_{k=0}^{n} A_n^k = \sum_{k=0}^{n} k! \binom{n}{k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!} = n! \sum_{k=0}^{n} \frac{1}{k!} \equiv ¡\, n$$

This number is obviously larger than the number of permutations of n distinct objects given by $n!$. Hence, a supfactorial, $¡\, n$, notation has been suggested. It is easy to check that $¡\, n$ satisfies the following recurrence relation,

$$¡\, n = n \cdot ¡\, (n-1) + 1$$

For very large $n \gg 1$, the supfactorial is nearly a constant times the factorial, $¡\, n \approx e \cdot n!$

Exercise. How many possible passwords can be composed using an alphabet of $n = 26$ letters, if a password is required to have at least 8 characters and have no repeating characters? Answer: $a_{26} - a_7 = 26! \sum_{k=8}^{26} \frac{1}{k!}$.

A (complete) **derangement** is a permutation of the elements of a set of distinct objects such that none of the objects appear in their original position. The number of derangements of a set of n distinct objects (or permutations of n distinct objects with no rencontres, or permutations with no fixed point) is smaller than $n!$ and is called the subfactorial, $!\,n$. It can be obtained by using the inclusion-exclusion principle. The universal set of permutations P has $n!$ elements. Denote P_1 the subset of permutations that keep element 1 in its place, P_2 those that keep element 2 in its place, P_k that keep element k in its place, and so on. The set of permutations that keep at least 1 element in its original place is then, $P_{>1} = P_1 \cup P_2 \cup P_3 \ldots \cup P_n$. The number of derangement is given by the number of elements complementing this set to P,

$$d_n = \left| P - \sum_{i=1}^{n} P_i \right| =$$

$$n! - \sum_{i=1}^{n} |P_i| + \sum_{1 \le i \le j \le n}^{n} |P_i P_j| - \sum_{1 \le i \le j \le k \le n}^{n} |P_i P_j P_k| \ldots + (-1)^n |P_1 \ldots P_n|$$

Using the fact that $|P_i|, |P_i P_j|, |P_i P_j P_l|, \ldots$ are equal to $(n-1)!, (n-2)!, (n-3)!, \ldots$, correspondingly, for every choice of $i, \{i,j\}, \{i,j,l\}, \ldots$ and there are $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots \binom{n}{k}, \ldots$ such choices, respectively, we obtain,

$$d_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \equiv\, !n$$

The number of derangements also obeys the following recursion relations,

$d_n = n \cdot d_{n-1} + (-1)^n$, or, $!n = n \cdot !(n-1) + (-1)^n$, and,

$d_n = (n-1) \cdot (d_{n-1} + d_{n-2})$, or, $!n = (n-1) \cdot \big(!(n-1) + !(n-2)\big)$.

Note that the latter recursion formula also holds for $n!$; for very large $n \gg 1$, the subfactorial is nearly a factorial divided by a constant, $!n \approx \frac{n!}{e}$. Starting with $n = 0$, the numbers of derangements of n are,

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, …

Exercise. A group of n men enter a restaurant and check their hats. The hat-checker is absent minded, and upon leaving, redistributes the hats back to the men at random. What is the probability, P_n, that no man gets his correct hat?

This is the old hats problem, which goes by many names. It was originally proposed by French mathematician P. R. de Montmort in 1708, and solved by him in 1713. At about the same time it was also solved by Nicholas Bernoulli using inclusion-exclusion principle.

An alternative solution is to devise a recurrence by noting that for a full derangement, every of n men should get somebody else's hat. Assume man x got the hat of man y. Assuming that man y got the hat of man x, there are d_{n-2} such possible derangements. However, we also must account for the possibility that man y, whose hat went to man x, did not get "his" hat of man x in return. This gets us to the situation of the full derangement for $n-1$ men. Adding the two possibilities and multiplying with $n-1$ possible choices of man y we obtain, $d_n = (n-1)(d_{n-1} + d_{n-2})$. For the probability we then obtain, or, $P_n = \frac{d_n}{n!} = P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$, wherefrom the expression for the derangement probability can be derived by setting up a "telescoping" sum, using $(P_1 = 0, P_0 = 1)$,

$$P_n - P_{n-1} = -\frac{1}{n}(P_{n-1} - P_{n-2}) = \frac{(-1)^2}{n(n-1)}(P_{n-2} - P_{n-3}) = \cdots$$
$$= \frac{(-1)^{n-1}}{n(n-1)\ldots .2}(P_1 - P_0) = \frac{(-1)^n}{n!}$$

If some, but not necessarily all, of the items are not in their original ordered positions, the configuration can be referred to as a partial derangement. The number of partial derangements with k fixed points (rencontres) is,

$$d_{n,k} = \binom{n}{k} d_{n-k} = \binom{n}{k} \sum_{p=0}^{k} \frac{(-1)^p}{p!}$$

Here is the beginning of this array.

n/k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	0	1					
3	2	3	0	1				
4	9	8	6	0	1			
5	44	45	20	10	0	1		
6	265	264	135	40	15	0	1	
7	1854	1855	924	315	70	21	0	1

<u>Some homework problems</u>.

Exercise. Verify the following recurrences for the number of arrangements, $a_n = n! \sum_{k=0}^{n} \frac{1}{k!} \equiv_{\text{¡}} n$,

$$a_n = n! \sum_{k=0}^{n} \frac{1}{k!} = n! \sum_{k=0}^{n-1} \frac{1}{k!} + \frac{n!}{n!} = n \cdot a_{n-1} + 1$$

Solution.

$$a_n = n! \sum_{k=0}^{n} \frac{1}{k!} = n! \sum_{k=0}^{n-1} \frac{1}{k!} + 1 = n \cdot (n-1)! \sum_{k=0}^{n-1} \frac{1}{k!} + 1 =$$
$$(n-1+1) \cdot (n-1)! \sum_{k=0}^{n-1} \frac{1}{k!} + 1 = (n-1) \cdot (n-1)! \cdot \sum_{k=0}^{n-1} \frac{1}{k!} +$$
$$(n-1)! \cdot \left(\sum_{k=0}^{n-2} \frac{1}{k!} + \frac{1}{(n-1)!} \right) + 1 = (n-1) \cdot (a_{n-1} + a_{n-2}) + 2$$

Homework for January 7, 2018.

1. Using the inclusion-exclusion principle, find how many natural numbers $n < 100$ are not divisible by 3, 5 or 7.

Solution. For $n < 100$, there are 33 divisible by 3, $|A_3| = 33$, 19 divisible by 5, $|A_5| = 19$, 14 numbers divisible by 7, $|A_7| = 14$. Also, there are 6 numbers divisible by $3 \cdot 5 = 15$, 4 divisible by $3 \cdot 7 = 21$, 2 divisible by $5 \cdot 7 = 35$, and none divisible by $3 \cdot 5 \cdot 7 = 105$. Hence, the answer is $99 - |A_3 + A_5 + A_7| = 99 - |A_3| - |A_5| - |A_7| + |A_{3\cdot 5}| - |A_{3\cdot 7}| - |A_{5\cdot 7}| = 99 - (33 + 19 + 14 - 6 - 4 - 2 + 0) = 99 - 54 = 45$.

2. Four letters a, b, c, d, are written down in random order. Using the inclusion-exclusion principle, find probability that at least one letter will occupy its alphabetically ordered place? What is the probability for five letters?
3. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.
4. Secretary prepared 5 different letters to be sent to 5 different addresses. For each letter, she prepared an envelope with its correct address. If the 5 letters are to be put into the 5 envelopes at random, what is the probability that
    a. no letter will be put into the envelope with its correct address?

   b. only 1 letter will be put into the envelope with its correct address?
   c. only 2 letters will be put into the envelope with its correct address?
   d. only 3 letters will be put into the envelope with its correct address?
   e. only 4 letters will be put into the envelope with its correct address?
   f. all 5 letters will be put into the envelope with its correct address?
5. Among 24 students in a class, 14 study mathematics, 10 study science, and 8 study French. Also, 6 study mathematics and science, 5 study mathematics and French, and 4 study science and French. We know that 3 students study all three subjects. How many of these students study none of the three subjects?
6. In a survey on the students' chewing gum preferences, it was found that
   a. 20 like juicy fruit.
   b. 25 like spearmint.
   c. 33 like watermelon.
   d. 12 like spearmint and juicy fruit.
   e. 16 like juicy fruit and watermelon.
   f. 20 like spearmint and watermelon.
   g. 5 like all three flavors.
   h. 4 like none.

   How many students were surveyed?

7. * If 9 dies are rolled, what is the probability that all 6 numbers appear?

Solution. The universal set I consists of 6^9 outcomes of rolling six dies. Denote A_n set of outcomes where one of the six numbers, $n = 1,2, \ldots ,6$, does not appear (A_1 is a set of outcomes where number 1 does not appear, and so on). There are six ways of choosing which number does not appear, so there are six such sets, $A_1, A_2, \ldots , A_6$. For every number, there are 5^9 outcomes where it does not appear, so $\forall n, |\, A_n| = 5^9$. We thus have $\binom{1}{6} \cdot 5^9$ outcomes where one of the digits does not appear. However, this over-counts by counting twice the outcomes where two digits do not appear, i.e. counting twice all possible pairwise intersections, $A_1 \cap\ A_2, A_1 \cap\ A_3, A_2 \cap\ A_3, \ldots$. There are $\binom{2}{6}$ such pairwise intersections and each contains 4^9 outcomes where only 4 numbers appear. Now, however, we have over-subtracted the $\binom{3}{6}$ triple intersections, each containing 3^9 outcomes where only 3 digits appear, and so on. Then, the number of outcomes where one of the digits does not appear is, $|A| = \binom{1}{6} \cdot 5^9 - \binom{2}{6} \cdot 4^9 + \binom{3}{6} \cdot 3^9 - \binom{4}{6} \cdot 2^9 + \binom{5}{6} \cdot 1^9$. The probability is obtained by dividing by $|I| = 6^9$, $P = |\mathrm{A}|/|\mathrm{I}|$. This is the same as using inclusion-exclusion principle. We need to find $|A_1 \cup A_2 \cup \ldots \cup A_6|$,

$$|A| = |A_1 + A_2 + \cdots + A_6| = \sum_{i=1}^{6} |A_i| - \sum_{1 \le i \le j \le 6}^{6} \left|A_i A_j\right| + \sum_{1 \le i \le j \le k \le 6}^{6} \left|A_i A_j A_k\right| - \cdots$$

8. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?