# MATH 8: HANDOUT 10
# LOGIC 5: PROOFS CONTINUED

## COMMONLY USED LAWS OF LOGIC

- Given $A \implies B$ and $A$, we can conclude $B$ *(Modus Ponens)*
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, $C$ is always true! It only means that **if** $A$ is true, then so is $C$.]
- Given $A \wedge B$, we can conclude $A$ (and we can also conclude $B$)
- Given $A \vee B$ and $\neg B$, we can conclude $A$
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ *(Modus Tollens)*
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ *(De Morgan Law)*
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ *(Law of contrapositive)*

**Note:** it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

## COMMON METHODS OF PROOF

**Proof by cases.**

*Example:* Prove that for any integer $n$, the number $n(n+1)$ is even.

*Proof.* If $n$ is integer, it is even or odd. If $n$ is even, then $n(n+1)$ is even (a multiple of even is always even). If $n$ is odd, then $n+1$ is even and thus $n(n+1)$ is even by same reasoning.

Thus, in all cases $n(n+1)$ is even. □

*General scheme:*

Given

$$A_1 \vee A_2$$
$$A_1 \implies B$$
$$A_2 \implies B$$

we can conclude that $B$ is true.

You can have more than two cases.

**Note:** it is important to verify that the cases you consider cover all possibilities (i.e. that at least one of the statements $A_1$, $A_2$ is always true).

**Conditional proof.**

*Example:* Prove that if $n$ is even, then $n^2$ is even.

*Proof.* Assume that $n$ is even. Then $n^2 = n * n$ is also even, since a multiple of even is even. □

*General scheme*

To prove $A \implies B$, we can

- Assume $A$
- Give a proof of $B$ (in the proof, we can use that $A$ is true).

This proves $A \implies B$ (without any assumptions).

**Proof by contradiction.**

*Example:* Prove that if $x$ is a real root of polynomial $p(x) = 10x^3 + 2x + 15$, then $x$ must be negative.

*Proof.* Assume that $x$ is not negative, i.e. $x \geq 0$. Then $p(x) = 10x^3 + 2x + 15 \geq 15$, which contradicts the fact that $x$ is a root of $p(x)$. Thus, our assumption can not be true, so $x$ must be negative. □

*General scheme*

To prove that $A$ is true, assume $A$ is false, and derive a contradiction. This proves that $A$ must be true.

## PROBLEMS

**1.** The following statement is sometimes written on highway trucks:

*If you can't see my windows, I can't see you.*

Can you write an equivalent statement without using word "not" (or its variations such as "can't").

**2.** Consider the following statement:

*You can't be happy unless you have a clear conscience.*

Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter $H$ for "you are happy" and $C$ for "you have a clear conscience".

Note: proving this statement is not part of the assignment :).

**3.** Here is another one of Lewis Carroll's puzzles. As before, (a) write the obvious conclusion from given statements; and (b) justify the conclusion, by writing a chain of arguments which leads to it.

- No one subscribes to the *Times*, unless he is well educated.
- No hedgehogs can read.
- Those who cannot read are not well educated.

It may be helpful to write each of these as a statement about some particular being $X$, e.g. "If $X$ is a hedgehog, then $X$ can't read."

**4.** Prove that for any integer number $n$, the number $n(n+1)(2n+1)$ is divisible by 3. Is it true that such a number must also be divisible by 6?

You can use without proof the fact that any integer can be written in one of the forms $n = 3k$ or $n = 3k+1$ or $n = 3k+2$, for some integer $k$.

**5.** You are given the following statements:

$$A \wedge B \implies C$$
$$B \vee D$$
$$C \vee \neg D$$

Using this, prove $A \implies C$.

**6.** A function $f(x)$ is called *monotonic* if $(x_1 < x_2) \implies (f(x_1) < f(x_2))$. Prove that a monotonic function can't have more than one root. *[Hint: use assume that it has at least two distinct roots and derive a contradiction.]*

**7.** Prove by contradiction that there does not exist a smallest positive real number.