

MATH 7: HANDOUT 8
BINOMIAL COEFFICIENTS FORMULA. BINOMIAL PROBABILITIES

BINOMIAL COEFFICIENTS: SUMMARY

We discussed binomial coefficients, and saw that they provide answers to the following questions:

$\binom{n}{k}$ = the number of paths on the chessboard going k units up and $n - k$ to the right
= the number of words that can be written using k ones and $n - k$ zeroes
= the number of ways to choose k items out of n (order doesn't matter)

FORMULA FOR BINOMIAL COEFFICIENTS

It turns out that there is an explicit formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Compare it with the number of ways of choosing k items out of n when the order matters:

$${}_n P_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

For example, there are $5 \cdot 4 = 20$ ways to choose to items out of 5 if the order matters, and $\frac{5 \cdot 4}{2} = 10$ if the order doesn't matter.

BINOMIAL PROBABILITIES

These numbers are also useful in calculating probabilities. Imagine that we have some event that happens with probability p ("success") and does not happen with probability $q = 1 - p$ ("failure"). Then the probability of getting k successes in n trials is

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k}, \text{ where}$$

- p — probability of success in one try;
- $q = 1 - p$ — probability of failure in one try;
- n — number of trials;
- k — number of successes;
- $n - k$ — number of failures.

Example: You roll a die 100 times. What is the probability of getting a 6 exactly 20 times?

Solution: Here we have: $n = 100$, $k = 20$, $p = 1/6$, $q = 5/6$. Then

$$P = \binom{100}{20} \cdot \left(\frac{1}{6}\right)^{20} \left(\frac{5}{6}\right)^{80}$$

HOMWORK

In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use Pascal triangle: $\binom{n}{k}$ is the k -th element in the n -th row of the Pascal triangle, counting from 0.

- (a) There are 15 students in a soccer club. The coach needs to select 11 of them to form the team for a match against another club. How many possibilities does he have?
 - (b) There are 15 students in a soccer club. The coach needs to select a goalkeeper and 10 players to form the team for a match against another club. How many possibilities does he have?
(The difference between two parts is that in the first case, the coach needs to select 11 players — no need to specify their positions. In the second part, he needs to select 11 players and specify which of them will be the goalkeeper.)
- In one of the lotteries run by New York State, “Sweet Million”, they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win \$1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]
 - (a) How many ways are there to choose 6 numbers out of 40?
 - (b) What are your chances of winning?
 - (c) If a lottery ticket cost \$1, how much money does New York State make for each ticket sold (on average)?
Bonus question: find online the rules for another NY lottery, “Mega Millions”, and analyze your chances to win.
- In poker, players are drawing “hands” (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
 - (a) How many possible hands are there?
 - (b) What are your chances of drawing a hand in which all cards are spades?
 - (c) What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]
 - (d) What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]
- A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$.
 - (a) The hunter makes 5 shots. What is the probability that she misses all five?
 - (b) What is the probability that out of 5 shots, she will hit a duck at least once? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)
 - (c) What is the probability that out of 5 shots, she will hit exactly once? Will this probability double if she makes 10 shots?
 - (d) What is the probability that out of 5 shots, she will hit a duck exactly three times? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)
 - (e) What is the probability that she hits a duck half times or more if she fires 5 times (that is, 3, 4, or 5 hits)? What about if she fires 10 times (that is 5, 6, 7, 8, 9, or 10 hits)?
 - (f) What is the most likely number of hits out of 5 shots? And out of 10 shots?
- At a fair, they offer you to play the following game: you are tossing small balls in a large crate full of empty bottles; if at least one of the balls lands inside a bottle, you win a stuffed toy (worth about \$5). Unfortunately, it is really impossible to aim, so the game is just a matter of luck (or probability theory): every ball you toss has a 20% probability of landing inside the bottle.
 - (a) If you are given three balls, what is the probability that all three will be hits? That all three will be misses? That at least one will be a hit?
 - (b) Same questions for five balls.
 - (c) What about seven balls?
 - (d) How much should the organizers charge for 3 balls to break even? What about for 5 balls?