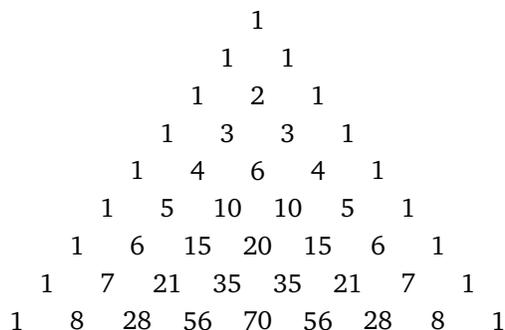


MATH 7: HANDOUT 7
PASCAL TRIANGLE AND ITS APPLICATIONS

PASCAL TRIANGLE

Recall the Pascal triangle:



Every entry in this triangle is obtained as the sum of two entries above it. The k -th entry in n -th line is denoted by $\binom{n}{k}$, or by $\binom{n}{n-k}$. Note that both n and k are counted from 0, not from 1: for example, $\binom{6}{2} = 15$.

In the previous handout we saw that these numbers appear in a problem about counting paths from the lower left corner of the board to the upper right corner. We observed the following:

$$\binom{n}{k} = \text{the number of paths on the chessboard going } k \text{ units up and } n - k \text{ units to the right}$$

For example, the number of paths that go to the upper right corner of a 6×6 board is equal to $\binom{10}{5}$, as each such path must have 5 steps to the right and 5 steps up. Now let us think about other applications of these numbers.

Words with 1s and 0s: Each on the board going up and to the right can be written as a sequence of steps, R for the step to the right, an U as a step up. For example, a path RRRRRUUUUU will go five step to the right and five steps up, eventually ending in the upper right corner of a 6×6 board. There is a correspondence between paths of length n and strings of length n consisting of Rs and Us only. Now let us switch Rs to 0s and Us to 1s. Now, we already know that $\binom{n}{k}$ is a number of paths going k units up and $n - k$ units to the right, which corresponds to words of length n with k Us and $n - k$ Rs, which is the same as a number of strings of length n with k 1s and $n - k$ 0s. We have the following result:

$$\binom{n}{k} = \text{the number of words that can be written using } k \text{ ones and } n - k \text{ zeroes}$$

Combinations: Now, let us consider all words on length n with k ones. The number of such words is $\binom{n}{k}$, as we showed above. Consider now a set on n elements, and let's number them from 1 to n . Now for each string on length n with 0s and 1s, we can select those elements that corresponds to 1s and omit those elements that correspond to 0s. That way, we will get a subset of size k . This way, we get another property of binomial coefficients:

$$\binom{n}{k} = \text{the number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}$$

To summarize, this is what we got:

$$\binom{n}{k} = \text{the number of paths on the chessboard going } k \text{ units up and } n - k \text{ to the right}$$
$$= \text{the number of words that can be written using } k \text{ ones and } n - k \text{ zeroes}$$
$$= \text{the number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}$$

HOMework

In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use Pascal triangle: $\binom{n}{k}$ is the k -th element in the n -th row of the Pascal triangle, counting from 0.

1. How many “words” of length 5 one can write using only letters U and R, namely 3 U’s and 2 R’s? What if you have 5 U’s and 3 R’s? [Hint: each such “word” can describe a path on the chessboard, U for up and R for right. . .]
2. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?
3. If we toss a coin 10 times, what is the probability that all will will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?
4. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub — and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will move
 - (a) 10 steps north
 - (b) 10 steps south
 - (c) 4 steps north
 - (d) will come back to the starting position
5. If you have a group of 4 people, and you need to choose one one to go to a competition, how many ways are there to do it? if you need to choose 2? if you need to choose 3?
6. How many ways are there to select 5 students from a group of 12?
7. In a meeting of 25 people, each much shake hands with each other. How many handshakes are there altogether?
8.
 - (a) An artist has 12 paintings. He needs to choose 4 paintings to include in an art show. How many ways are there of doing this?
 - (b) The same artist now needs to choose 4 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).