MATH 7: HANDOUT 1
ALGEBRAIC EXPRESSIONS AND IDENTITIES

Today we discussed how one works with algebraic expressions, i.e. expressions containing variables, such as $2(x+1)-3$. You probably already know most of these identities, but it is important to get them all together. In particular, we discussed the following useful formulas about exponents, radicals, and the basic algebraic identities.

EXPONENTS LAWS

If a is a real number, n is a positive integer, we define $a^n = \underbrace{a \times a \times \cdots \times a}_{n\text{-times}}$

$$a^0 = 1$$
$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(ab)^n = a^n b^n$$
$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
$$a^{-n} = \frac{1}{a^n}$$
$$(a^m)^n = a^{mn}$$

Why do we assume that $a^0 = 1$? We can see it from the following argument:

$$a^1 = a^{1+0} = a^1 a^0$$

Therefore, we must have the following equality (if we want to maintain the properties of exponents – and we do!):

$$a = a \cdot a^0$$
$$a^0 = 1$$

Now, why do we assume that $a^{-n} = \frac{1}{a^n}$? We can see it from the following argument:

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1.$$

Therefore, dividing both parts of the equation by a^n, we get:

$$a^{-n} = \frac{1}{a^n}.$$

RADICALS

Now, what should the fractional powers be? Let us figure out what $a^{1/2}$ is. We will use similar logic as above:

$$a^{1/2} \cdot a^{1/2} = a^1 = a.$$

Therefore, $(a^{1/2})^2 = a$; from here, we can see that $a^{1/2} = \sqrt{a}$. Similarly, we can find that $a^{1/n} = \sqrt[n]{a}$. In general, we have the following properties:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, n \neq 0$$
$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

Main Algebraic Identities

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a-b)(a+b)$$

All of these formulas can be proven by performing multiplication, e.g. $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$. You can verify the others by yourself.

Homework

Please try to do as many of the problems below as you can, and submit your completed solutions through Google Classroom. Some of these problems are similar to those we have discussed in class; some are new. It is OK if you can not solve some problem — but do not give up before making an effort, maybe putting the problem away and coming back to it later — which means you have to start the homework early.

You solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Without a calculator, compute

$$19999 \cdot 20001$$

Is there a shorter way of doing it than the straightforward multiplication?

2. Expand

(a) $2x(a+2b+3c)$
(b) $-3y(a-ay+by)$
(c) $(a^2+2a+1)(a+1)$
(d) $(b^2-2b+1)(b-1)$
(e) $(4x-7y)(4x+7y)$
(f) $(6x^2-y)(7x^2-2x-5)$

3. Factor (i.e., write as a product) the following expressions:

(a) $ac+ab$
(b) x^2+3x^3
(c) $x^2-2x-yx+2y$
(d) $4x^2-4x+1$
(e) $4x^2+16x+2xy+8y$
(f) $x^2(x+4)+5(x+4)$
(g) $100x^8y^2-16x^4y^6$
(h) $a^2+4ab+4b^2$
(i) a^2-2a+1
(j) x^2-7 [Hint: $7=(\sqrt{7})^2$.]
(k) a^4-b^4 [Hint: $a^4=(a^2)^2$.]

4. John takes 15 min to walk from school to the bus station. Jim takes 20 min to walk from the school to the bus station. If the difference in their speeds is 2 km/h, how far is the station from the school?

5. Simplify:

(a) $\dfrac{1}{x+1} - \dfrac{1}{x-1}$ (b) $\left(1+\dfrac{1}{x}\right) \div (x+1)$ (c) $\left(1+\dfrac{1}{x}\right) \div \left(1-\dfrac{1}{x}\right)$