Homework 14: the quadratic formula

HW14 is Due January 24; submit to Google classroom 15 minutes before the class time.

1. Quadratic equation in a standard form.

Today we discussed how one solves quadratic equation, starting from the **standard form**: $ax^2 + bx + c = 0$
A quadratic equation could have no solution, one solution, or two solutions depending on the coefficients a, b, and c.

We could solve such an equation by presenting it in a **factored form**: $(x - x_1)(x - x_2) = 0$, where x_1 and x_2 are the solutions of the equation, also known as *roots.* The factored form will also help us find a general formula for solving any quadratic equation using the coefficients a, b, c.

2. Solving the incomplete quadratic equation by factorizing.

- When $c = 0$, $ax^2 + bx = 0$

To solve, factorize as $x(ax + b) = 0$ and the two terms in the product to be equal to zero. The two roots are $x_1 = 0$ and $x_2 = -b/a$

- When $b = 0$, $ax^2 + c = 0$

If c < 0, factorize the equation using the formula for fast multiplication $a^2 - b^2 = (a - b)(a + b)$. (*)
For example, $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0$. Setting each term in the product to zero gives solutions of $+5$ and -5.

If c > 0, there are no real solutions. An easy way to see this is to solve directly for x: $x^2 + 25 = 0 \Rightarrow x^2 = -5^2$; No number squared is equal to a negative number!

2. Solving the complete quadratic equation

- By completing the square

"Completing the square" works by using the formulas for fast multiplication $(a \pm b)^2 = a^2 \pm 2ab + b^2$ (*)
Here is an example how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x + 3)^2 - 7 = (x + 3)^2 - (\sqrt{7})^2 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

Thus, $x^2+6x+2 = 0$ if and only if $(x+3+\sqrt{7}) = 0$, which gives $x = -3-\sqrt{7}$, or $(x+3-\sqrt{7}) = 0$, which gives $x = -3+\sqrt{7}$.

- By using the quadratic formula

Completing the square works in general for any quadratic equation in a standard form
If *a* = 1, then:

$$x^2 + bx + c = x^2 + 2\frac{b}{2}x + c = \left(x^2 + 2\frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{2^2} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{2^2} \quad \text{eq (1)}$$

Thus $x^2 + bx + c = 0$ is equivalent to: $\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$

If $a \neq 1$, then: $ax^2 + bx + c = 0$ is equivalent to: $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$, where $\boldsymbol{D = b^2 - 4ac}$
The determinant D determines the number of solutions. D < 0, there are no real solutions; if D = 0, there is one solution,

if $D > 0$, the solutions are:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{D}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{D}}{2a} \quad \text{eq (2)}$$

(*) The parameters a and b in the formulas for fast multiplication $a^2 - b^2 = (a - b)(a + b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$ are not the same as the coefficients a, b, and c used in the standard form of the quadratic equation!

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

Note: Use the formulas for fast multiplication $a^2 - b^2 = (a-b)(a+b),\ (a \pm b)^2 = a^2 \pm 2ab + b^2$.

1. Complete the square and solve the quadratic equations: using $(a \pm b)^2 = a^2 \pm 2ab + b^2$ and then $a^2 - b^2 = (a-b)(a+b)$
   a. $x^2 - 2x - 3 = 0$
   b. $x^2 + 8x - 9 = 0$

2. Solve the following equations. Carefully think what method you will use and <u>write all steps</u> in your solution. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients a, b, c? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (2)?
   c. $x^2 - 5x + 5 = 0$
   d. $x^2 = 1 + x$
   e. $-4x^2 + 8x + 21 = 0$
   f. $2x(3 - x) = 1$
   g. $x^3 + 4x^2 - 45x = 0$
   h. $\frac{x}{x-2} = x - 1$

3. Indian mathematicians were aware of the quadratic formula for solving quadratic equations. Can you solve the following problem by the 9th century mathematician Mahavıra? (translated from original text)
   One-third of a herd of elephants and three times the square root of the remaining part of the herd were seen on a mountain slope; and in a lake was seen a male elephant along with three female elephants constituting the ultimate remainder. How many were the elephants here?

4. In the 12th century, Indian mathematician Bhaskara formulated the following problem. Solve it! (translated from original text)
   Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave. The one remaining monkey was climbing up a tree. What is the total number of monkeys?

5. This problem requires that you carefully check your work and think:
   a. <u>Use formula (1)</u> to prove that for any x, $x^2 + bx + c \geq -D/4$, with equality only when $x = -b/2$.
   b. Find the minimal possible value of the expression $x^2 + 4x + 2$ [Hint: use part a) or complete the square]
   c. Given a number *a* > 0, find the maximal possible value of the expression $x(a - x)$ (the answer will depend on the value or values of a. In this case, a is called a *parameter*).

6. If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2} = 7$ and $x^3 + \frac{1}{x^3}$ [Hint: try completing the square, completing the cube …]

7. (*) Consider the sequence $x_1 = 1,\ x_2 = \frac{x_1}{2} + \frac{1}{x_1},\quad x_3 = \frac{x_2}{2} + \frac{1}{x_2}\ \ldots.$

   Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you ***guess*** this value? [Hint: solve equation $x = \frac{x}{2} + \frac{1}{x}$]

(*) The parameters a and b in the formulas for fast multiplication $a^2 - b^2 = (a-b)(a+b),\ (a \pm b)^2 = a^2 \pm 2ab + b^2$ <u>are not the same</u> as the coefficients a, b, and c used in the standard form of the quadratic equation!