MATH 6A/D: HOMEWORK 22
DEADLINE: FRIDAY, MARCH 26TH, 2021

In our last class we reviewed a few fundamental notions of coordinate geometry: slope-intercept and standard forms of linear equations, distance between points, the equation of a circle, the midpoint of a segments, parallel and perpendicular lines. Next time we'll play either a game of mathematical hockey or a game of mathematical race. Please familiarize yourselves with the rules of these games before the class (see the section "Math Hockey and Math Race" below). I'll choose which one we'll play closer to the weekend. If you have any preferences, please let me know in the comments, I'll do my best to take them into account.

Parallel and Perpendicular Lines

Parallel lines have the same slopes so the linear equations corresponding to a pair of parallel lines in their slope-intercept form look like $y = mx + b_1$ and $y = mx + b_2$.

Perpendicular lines have slopes that are negative reciprocals of each other (for example, the negative reciprocal of $\frac{3}{4}$ is $-\frac{4}{3}$). The linear equations corresponding to a pair of perpendicular lines in the their slope-intercept form look like $y = mx + b_1$ and $y = -\frac{1}{m}x + b_2$.

Equation of a Circle

Since the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the following formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

the equation of the circle with the center $M(x_0, y_0)$ and radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

This equation means, that points (x, y) should be at distance r from the given point $M(x_0, y_0)$.

For example, the equation of a circle with the center in the origin $(0, 0)$ and radius 5 that we discussed in class is

$$x^2 + y^2 = 25.$$

Methods of Solving Systems of Linear Equations

Even though we didn't have time to formally discuss methods of solving systems of linear equations in class, I decided to give you a chance to work on two methods on your own using the notes below. If needed, we'll cover it next time before the game.

1. Elimination method

(a) Simplify both equations.

$$\begin{aligned} &-9y + 4x - 20 = 0 \\ &-7y + 16x - 80 = 0 \end{aligned}$$

(b) Look for coefficients in front of one of the unknowns, x or y, which are the same (in our case, they are not the same, see the next step).

(c) If the coefficients are different try to make them the same by multiplying one or both equations by a number.

$$\begin{aligned} &-36y + 16x - 80 = 0 \\ &-7y + 16x - 80 = 0 \end{aligned}$$

(d) Add/subtract the two equations so this unknown cancels out (in our case we subtract the first one from the second one). Now, you have one equation with one unknown, solve it.

$$
\begin{aligned}
&29y = 0 \\
&y = 0
\end{aligned}
$$

(e) Go back to one of the two equations in your system, substitute the unknown you just found, find the second unknown.

$$
\begin{aligned}
&0 + 4x - 20 = 0 \\
&x = 5
\end{aligned}
$$

(f) Check that the values you found satisfy both equations by plugging them in.

$$
\begin{aligned}
&-9 \times 0 + 4 \times 5 - 20 = 0 \\
&-7 \times 0 + 16 \times 5 - 80 = 0
\end{aligned}
$$

2. Substitution method

(a) Simplify both equations.

$$
\begin{aligned}
7x + 10y &= 36 \\
-2x + y &= 9
\end{aligned}
$$

(b) From one of the 2 equations, express one of the unknowns (for example, y) in terms of the other one (y = . . .) .

$$y = 9 + 2x$$

(c) Substitute the obtained expression in the other equation - you have an equation with one unknown (linear equation for x).

$$
\begin{aligned}
&7x + 10(9 + 2x) = 36 \\
&27x = -54
\end{aligned}
$$

(d) Solve this equation (find x).

$$x = -2$$

(e) Substitute the value for the second unknown (the x-value) back in the first equation (in y = . . .).

$$
\begin{aligned}
&y = 9 + 2 \times (-2) \\
&y = 5
\end{aligned}
$$

(f) Check that the values you found satisfy both equations by plugging them in.

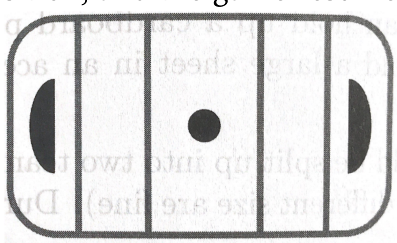
$$
\begin{aligned}
7 \times (-2) + 10 \times 5 &= 36 \\
-2 \times (-2) + 5 &= 9
\end{aligned}
$$

Mathematical Hockey Rules

The goal of the game is the same as of the actual hockey: to score goals by shooting the puck into the opponent's goal net. However, in a game of math hockey the ice rink, the puck and the goal nets are not real: they are drawn on the board. The puck is driven by the success of a team in solving math problems that are thrown into the game by a referee (me).

The game is played in short problem-solving rounds. On every round, each team sends up a delegate of its choice, subject to the rule that a full rotation of delegates should happen within every presentation cycle. (For example, if a team has 5 students in it, everybody should present once withing the first 5 rounds, twice within the first 10 rounds, three times withing the first 15 rounds and so on.). The referee (me), throws in a challenging problem for the round. Both delegates start working on the problem, without help from their teams. The first delegate to solve the problem announces his or her answer. If the answer is correct, this player's team wins the round; otherwise, the opposite team wins, without having to present its answer.

The team that won the round advances the puck one zone closer to the opponents goal posts. Once a puck gets into the goal zone, the goal is scored. The puck is returned to the central position, and the game resumes. Here is how the rink is divided into zones:



It can happen that neither delegate can come up with an answer on a given round. The referee has two choices: either to discard the question, or allow lifelines (help from the team). I like the option of lifelines more because it'll keep the rest of the team members more engaged. The team that scores most points wins the game.

Mathematical Race Rules

Problem solving. During the competition, each team (2-4 players) solves its way through the same set of problems. The teams start working on the problems, presenting solutions as they go. The only restriction is that a team works on 4 problems at a time. Thus, at the start of the game, a team receives the first four problems. Whenever a team presents the correct solution to a problem it currently owns, it receives the next one from the list.

For example, suppose that a team currently has problems 1, 2, 3, and 4. If it presents a correct solution to problem 3, the team receives problem 5 in return. Next, if it presents problem 1, the team receives problem 6, etc.

A team has up to 3 attempts per problem; successful and failed attempts are marked at the score table (we can keep track of each team's score in Google Sheets).

Points. A team's final standing depends on the total number of problems solved, as well as on the longest stretch of correct answers. For every problem solved, the team receives 3 points. Also, for the longest stretch of correct answers, the team receives 1 extra point per answer.

PROBLEMS

1. Find the midpoint of the segment ST given S(-2, 8) and T(10, -4).
2. Determine the equation of the line that goes through points (6,1) and (3,-4) without graphing.

3. Using the property of the slopes of perpendicular lines determine if the segment through the origin and point A and the segment through the origin and point B are perpendicular:
(a) A(-3, -4), B(4, 3)
(b) A(8,9), B(18,-16)

4. Given points N(7,6) and M(7,-2)
(a) Write the equation of the line through M and perpendicular to MN
(b) Write the equation of the line through N and perpendicular to MN

5. Compute the area of the quadrilateral ABCD if A(0, 0), B(2, 3), C(-4,7) and D(-6, 4).

6. (a) Draw the graph of the equation $x^2 + y^2 - 1 = 0$.
(b) Draw the graph of the equation $(x - 3)^2 + (y - 3)^2 = 9$.
(c) Draw the graph of the equation $x^2 + y^2 = 0$.

7. Solve systems of linear equations by using the elimination method
(a)

$$\begin{aligned} 2x + y &= 8 \\ 3x + y &= 10 \end{aligned}$$

(b)

$$\begin{aligned} x - 3y &= 2 \\ x - 5y &= 2 \end{aligned}$$

(c)

$$\begin{aligned} 5y - 2x &= 1 \\ 15y - 3x &= -3 \end{aligned}$$

(d)

$$\begin{aligned} (2x - 3)(3y - 4) &= (2y - 5)(3x + 1) \\ 3(y + 2) - 2(x - 3) &= 16 \end{aligned}$$

8. Solve systems of linear equations by using the substitution method
(a)

$$\begin{aligned} x &= 5 \\ 20x + 5y &= 100 \end{aligned}$$

(b)

$$\begin{aligned} -8x + y &= -4 \\ -21x + 2y &= -13 \end{aligned}$$

(c)

$$\begin{aligned} 7x - 3y &= 27 \\ 5x - 6y &= 0 \end{aligned}$$

(d)

$$\begin{aligned} 2(x - 2) - 3(x + y) &= 3 \\ (x + 1)(y - 2) &= xy - 9 \end{aligned}$$

9. Find the shortest distance from the origin (0, 0) to the line given by the equation $y = -2x + 8$.[Hint: Find the equation of the line perpendicular to the line corresponding to the given equation and going through the origin. Solve the system of the two equations to find the coordinates of the point where they intersect (the pair (x,y) that satisfies both equations). Use the distance formula to calculate the distance from the origin to the intersection point.]

***10.** Come up with a number that ends with 17, is divisible by 17, and has the sum of its digits equal to 17.