Math 6a/d: Homework 18

Deadline: Friday, February 26, 2021

Divisibility

Definition

Last time we discussed that an unknown even integer n can be represented as $n=2\times k$, where k is another unknown integer. Similarly, an unknown odd integer n can be represented as $n=2\times k+1$, where k is another unknown integer.

General algebraic definition of divisibility: Let A and B be integers. We say that A is divisible by B ($A \mid B$) if it is possible to find an **integer** K such that $A=B\times K$. We can also express in writing the fact that A is not divisible by B ($A \nmid B$) as $A=B\times K+R$ where $0<R\leq(B-1)$.

In this class we also talked about two ways to visualize definition of divisibility:
Definition 1. An integer number A is divisible by an integer number B if it is possible to spilt A objects into several groups so that every group contains B objects (see the picture in the Jamboard).
Definition 2. An integer number A is divisible by an integer number B if it is possible to split A objects into B groups so that every group contains the same number of objects (see the picture in the Jamboard).

These two definitions are related since the number of objects in each group in the first definition is the same as the number of groups in the second one and equals to K from our algebraic definition.

Some properties of divisibility
1. Divisibility of a sum: if A and B are integers, $A \mid C$ and $B \mid C$, then $(A+B) \mid C$
2. Divisibility of a product: If A is an integer and $A \mid C$ and if B is any integer, then $A\times B \mid C$
3. Divisibility of a difference: if A and B are integers, $A \mid C$ and $B \mid C$, then $(A-B) \mid C$

Closely related to divisibility is prime factor decomposition (= prime factorization). For any number we take, whichever way we do the factorization, we always end up with the same set of prime factors so **the set of prime factors of a number is unique** (this statement is known as the Fundamental Theorem of Arithmetic). Some of you noticed that this set of prime factors is like the fingerprints of a number. I really like the analogy. From prime factorization we can draw conclusions about divisors of a number and its divisibility. Since prime factorization is really useful we need to have a way to find prime factors of a number. Divisibility rules are shortcuts that allow us to detect divisibility without performing the division.

Divisibility rules

Divisibility by 2: A number is divisible by 2 if it ends with 0, 2, 4, 6, or 8.
This rule works because any integer number can be represented as a sum of a number that ends with 0 and a one-digit number (for example, 134=130+4). Any number that ends with 0 is divisible by 10; therefore it is also divisible by 2. Thus , divisibility of the whole sum is determined by divisibility of the second term (the one-digit number). Since 0, 2, 4, 6, and 8 are the only one-digit number that are divisible by 2, a number must end with these digits in order to be divisible by 2.

Divisibility by 4: A number is divisible by 4 if its last digits form a number that is divisible by 4.
To justify this rule, first, we can observe that all integers that end with two zeroes are divisible by 4: 100, 200, 900, 100, 1100 etc. They do because they are multiples of 100 - which, in turn, is divisible by 4. The second useful observation is that an integer number can be split into the sum of a number that ends with two zeroes and two-digit number (for example, 1732=1700+32). Since a number that ends with two zeroes is divisible by 4, the divisibility of the second term defines divisibility by 4. This observation proves the rule.

Divisibility by 5: A number is divisible by 5 if it ends with 0 or 5.
Divisibility by 5 is explained along the same lines as divisibility by 2.

Divisibility by 3: A number is divisible by 3 if the sum of its digits is divisible by 3.

Divisibility by 9: A number is divisible by 9 if the sum of its digits is divisible by 9.

You'll be asked to justify the divisibility rules by 3 or by 9 in your homework. You are welcome to look it up on the Internet, watch here (https://www.youtube.com/watch?v=NehkLV77ITk and https://www.youtube.com/watch?v=XAzFGx3Ruig) or come up with your own proof.

Enjoy your homework!

Problems

1. Mister X, Mister Y and Lady Z are all inhabitants of the Knights and Knaves Island.

(a) Mister X says that whenever two integers are divisible by 7, their difference must be divisible by 7 as well. Is Mister X a knight or a knave? Justify your answer.

(b) Mister Y says that whenever the difference of two integers is divisible by 7, each number must be divisible by 7 as well. Is Mister Y a knight or a knave? Justify your answer.

(c) Lady Z states that whenever two integers A and B are each divisible by a third integer C, their difference must be divisible by C as well. Is Lady Z a knight or a knave? Justify your answer.

2. (a) Can you come up with four consecutive integer numbers that are not divisible by 4? Either present these numbers or explain why they do not exist.
(b) How about five consecutive integer numbers that are not divisible by 5?

3. There are three types of coins in the country of Sugarland: sugriks, tugriks and shmollars. Sugricks are the lowest denomination coins: both tugriks and shmollars are worth an integer number of sugriks. The price of a magic lollipop at the Sugarland's Sunday market is 11 sugriks. Little Aish has 4 sugriks, 1 tugrik, and 9 shmollars in her pocket. She uses all this money to pay exactly for several magic lollipops.

Little Ben has 15 tugriks, 12 tugriks and 9 shmollars with him. Prove that he has exact change to pay for several magic lollipops as well.

4. Two number have respective prime factor decomposition $2^2 \times 3 \times 7^3 \times 13$ and $2 \times 3^2 \times 7^2$,

(a) Is the first number divisible by the second?

(b) Is the product of these numbers divisible by 8? By 36? By 27? By 16? By 56?

5. Let p be a prime number greater than 2.

(a) Prove that at least one of the numbers $p-1$ and $p+1$ is divisible by 4.

(b) What about divisibility by 5?

6. Justify one of the two divisibility rules: divisibility by 3 or divisibility by 9 (see the notes section at the beginning of this homework).

7. Can an integer composed of only digits 4 be divisible by an integer composed of only digits 3? How about the other way around? For each question, either come up with a pair of numbers or explain why such a pair cannot be found.

8. Secret Service Agent 00X got access to a top-secret safe protected by a digital lock. The agent knows that the code for the lock has seven digits: some of these digits are 2's, others are 3's. He also knows that there are more 2's than 3's, and that this code is divisible both by 3 and by 4. Agent 00X has to open this lock on the first try - otherwise, the alarm will sound. Help the agent open the lock safely.

9. Little Michael came up with an encrypted problem:

$$AB - BA = 7.$$

(However, no matter how hard he tried, he couldn't solve it. Explain why. The same letter stands for the same digit, and different letters stand for different digits.).

10. A vicious giant captured 3 wise gnomes. He intends to eat the gnomes; however, he decides to play a game with them first. The giant shows the gnomes a box with 2 red hats and 3 black hats. Then the giant hands each gnome a piece of paper, turns off the lights, and puts 3 of the 5 hats on the heads of 3 gnomes. After the giant turns the lights on, each gnome is able to see the color of all the other gnomes' hats, but not the color of his own. The giant explains that each gnome with be given a chance to guess the color of his hat. If a gnome makes the right guess, he is free to go. If the guess is wrong, the giant will eat the gnome.

The guessing works as follows: the giant will wave his hand three times. At each wave, every gnome will have a choice: he can either write the color of his hat on the piece of paper and hand it to the giant, or do nothing. After each wave, the giant will read aloud the notes he had received. After three waves, the giant will release those gnomes who guessed the colors correctly.

The gnomes are not allowed to talk to each other or to communicate in any non-verbal manner. However, the gnomes are very wise (they all attended SchoolNova math course during their school years). Each gnome really wants to save himself and his friends. Show that all the gnomes have a way to guess the colors of their hats correctly.