

HANDOUT 11: ARITHMETIC SEQUENCES

1. ARITHMETIC SEQUENCE

Carl Friedrich Gauss (1777-1855) was a German mathematician who contributed in many fields of mathematics and is considered one of the greatest mathematicians of all times.

When he was a young boy, his class was given the problem of adding all integers from 1 to 100. The teacher thought that all students will be quite busy for some time! But the teacher was very surprised when young Gauss gave the answer in 5 minutes. He found an easy way to add all numbers in an arithmetic sequence.

$$\begin{array}{r} 1 + 2 + 3 + \dots + 99 + 100 \\ 100 + 99 + 98 + \dots + 2 + 1 \\ \hline \end{array}$$

$$101 + 101 + 101 + \dots + 101 + 101 = 101 \times 100 = 10100$$

$$\text{The sum of all numbers from 1 to 100 is } \frac{10100}{2} = 5050$$

A sequence of numbers (typically but not always integers) is an arithmetic sequence if the common difference between consecutive terms is the same number. In the example above the common difference is 1.

Let's consider the sequence: 1, 5, 9, 13, 17, ...

The first term in the sequence is 1, the second is 5, and so on. The common difference between consecutive terms is 4.

What is the n^{th} term? For example what is a_{100} ?

$$1$$

$$1 + 4 = 5$$

$$(1 + 4) + 4 = 1 + 2 \times 4 = 9$$

$$1 + 3 \times 4 = 13$$

....

$$1 + (99) \times 4 \text{ is the 100th term}$$

In general the n -th term of an arithmetic sequence is

$$a_n = a_1 + (n - 1)d$$

2. PROPERTY OF AN ARITHMETIC SEQUENCE

A property of an arithmetic sequence is that any term is the arithmetic mean of its neighbors.

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

3. SUM OF AN ARITHMETIC SEQUENCE

$$S = a_1 + a_2 + a_3 + \dots + a_n = n \cdot \frac{a_1 + a_n}{2}$$

To prove this, we write the sum in 2 ways, in increasing and decreasing order:

$$S = a_1 + a_2 + a_3 + \dots + a_n$$

$$S = a_n + a_{n-1} + a_{n-2} + \dots + a_1$$

Adding, we notice that $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$

$$2S = (a_1 + a_n)n \text{ and dividing by 2 gives us } S = n \cdot \frac{a_1 + a_n}{2}$$

4. HOMEWORK PROBLEMS

1. $1 + 2 + 3 + \dots + 100 + 101 =$
2. $1 + 3 + 5 + \dots + 99 =$
3. $11 + 12 + 13 + \dots + 101 =$
4. Write the first 5 terms of an arithmetic sequence if $a_1 = 7$ and common difference = 2.
5. What are the first 2 terms of the arithmetic sequence $a_1, a_2, -9, -2, 5, \dots$?
6. What is a_1 and what is the common difference for the following arithmetic sequence: $-10, -5, 0, 5, 10, \dots$? What is the 25th term?
7. $a_{10} = 131$ and the common difference $d = 12$. What is a_1 ?
8. Find the sum of the first 100 terms if $a_1 = -2$ and the common difference is 1.
9. Simplify the following expression:

$$\frac{\frac{2}{\frac{1}{1 - \frac{1}{3}} - 1}}{\frac{\frac{1}{2}}{\frac{2}{3} - \frac{1}{4}}}$$