MATH 6: HANDOUT 8

Factorials and permutations

Similar to our product rule from last time, when determining how many ways there are to make combination choices, we multiply the number of ways to make each choice, provided they are independent. Here are some examples:

- If I flip a coin three times and write down the three outcomes (heads or tails) in order, there are $2 * 2 * 2 = 8$ eight possible final results: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.
- If I have three logic statements that can each be true or false, then there will be $2 * 2 * 2 = 8$ rows in a truth table for these statements
- If there are eight volume settings for the left speaker and eight for the right speaker, then there are $8 * 8 = 64$ total stereo volume settings
- If I make a card that has one of four suits and one of thirteen numbers on it, then there are $4 * 13 = 52$ possible ways to make such a card
- If I have 7 choices of x-coordinate and 5 choices of y-coordinate, then there are 35 choices of (x, y) coordinate points (to see this, think about a 5×7 table or grid)
- If I have four friends and I want to choose one to stand to my right in my profile picture and one to stand to my left, there are $4 * 3 = 12$ ways to arrange this (4 ways to choose who stands on my right, and then 3 remaining people to choose who stands on my left)

In general, if we are choosing k objects from a collection of n so that a) order matters and b) no repetitions are allowed, then this is referred to as a *permutation* of k objects from the collection of n, the number of ways to make such a selection of permutations is called ${}_nP_k$, and

$${}_nP_k = n(n-1)\dots(n-k+1) \quad (k \text{ factors})$$

In particular, if we take $k = n$, it means that we are selecting one by one all n objects — so this gives the number of possible ways to order n objects:

$$n! = {}_nP_n = n(n-1)\cdot \dots \cdot 2 \cdot 1$$

We read $n!$ as "n factorial". By convention, $0! = 1$, similar to the way that $x^0 = 1$.

For example: there are 52! ways to mix the cards in the usual card deck.

Note that the number $n!$ grow very fast: $2! = 2$, $3! = 6$, $4! = 2 \cdot 3 \cdot 4 = 24$, $5! = 120$, $6! = 620$

Using factorials, we can give a simpler formual for ${}_kP_n$:

$${}_nP_k = \frac{n!}{(n-k)!}$$

For example:

$${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{6!}{2!}$$

Homework

1. You have 4 shirts and 4 ties colored red, yellow, blue and green. How many shirt and tie combinations are there if you refuse to wear a shirt and a tie of the same color?

2. A sly elementary school teacher decides to play favorites without telling anyone. If they have 15 students in their class, in how many ways can they choose a favorite student, a second favorite student, and a third favorite student?

3. Compute $\frac{5!}{4!}$, $5! - 4!$, ${}_3P_2$, ${}_3P_3$

4. (a) How many ways are there to draw 3 cards from a 52-card deck? (Order matters: drawing first king of spades, then queen of hearts is different from drawing them in opposite order).

(b) How many ways are there to draw 3 cards from a 52-card deck if after each drawing we record the card we got, then return the card to the deck and reshuffle the deck? (As before, order matters.)

(c) We draw 3 cards from a 52-card deck, and after each drawing we record the card we got, then return the card to the deck and reshuffle the deck. What is the probability that all 3 drawn cards are different?

5. Recall the

(a) About 1/6 of the citizens of Amphibian York City are newts. If we choose six citizens at random for a survey, what is the probability that all of them are newts? That none are newts? That at least one is a newt?

(b) In the Great Amphibian Parliament, there are six speakers of the species: Tandra the Toad, Dina the Dart Frog, William the Wood Frog, Sukarno the Salamander, Nassim the Newt, and Sibyl the Siren. Everyday when parliament meets, they sit in the front row of the room, which has exactly six seats. They want to sit in a different order each day, so that videos of the parliament's proceedings will indicate the date by looking at the front row. Can they devise such a system where they sit in a different order every day for a year? How about two years?

6. I want to make a necklace with six beads - I have one bead each in purple, blue, violet, and mauve, and two red beads. I care about the order in which the colors are arranged - how many ways are there to arrange the beads into an order for the necklace? (Assume that the two red beads are identical.)

***7.** A stubborn chemistry student refuses to learn the structure of the molecule caffeine. On their final exam, they are asked to fill in a diagram with which atom goes in which spot. The hydrogens are already filled in, but of the 14 remaining spots, the student has to fill in 8 carbon atoms, 4 nitrogen atoms, and 2 oxygen atoms. Assuming the student has no idea how chemical bonding works, what is the probability that they will fill in the correct arrangement?

8. (a) How many 5s are there in the prime factorization of the number 100! ? How many 2s?

*(b) In how many zeroes does the number 100! end?

9. Simplify: $(2^{10} \div (2^3)^3 + (2^2)^{2^2} + (3^8)^9 + ((3^2)^4)^9 - ((2^7)^3)^2) \div (1 + 2^6 + 4^3 + 3^{72} - 2^{41})$
(Hint: $2^7 = 2 \cdot 2^6 = 2^6 + 2^6 = 2^6 + 4^3$)