MATH 10
ASSIGNMENT 23: SUBGROUPS

APR 25, 2021

Definition. Let G be a group. A subgroup of G is a subset $H \subset G$ which is itself a group, with the same operation as in G. In other words, H must be

1. closed under multiplication: if $H_1, h_2 \in H$, then $h_1 h_2 \in H$
2. contain the group unit e
3. for any element $h \in H$, we have $h^{-1} \in H$.

Examples are given in problem 1 below.

Homework

1. Are these subgroups?
(a) $G = \mathbb{Z}$ (with operation of addition), $H = 5\mathbb{Z}$ =multiples of 5.
(b) $G = \mathbb{Z}$ (with operation of addition), $H = \{n = 5k + 1\}$.

2. Let G be a group, and let $a \in G$. Consider the set of all powers of a:

$$H = \{a^n \mid n \in \mathbb{Z}\} \subset G$$

(note that n can be negative).
(a) Show that H is a subgroup (this is called the subgroup generated by a). Subgroups of this form are also called *cyclic* subgroups.
(b) Describe explicitly the cyclic subgroup in $\mathbb{Z}_{10}$ generated by 2; by 3; by 6.

3. Let $H \subset G$ be a subgroup. For any element $g \in G$, define the subset

$$[g] = gH = \{gh, h \in H\}$$

Subsets of this form are called *cosets*. Note that two different elements can define the same coset.
(a) List all cosets in the case when $G = \mathbb{Z}, H = 5\mathbb{Z}$.
(b) Show that two elements x, x' are in the same coset gH iff $x' = xh$ for some $h \in H$.
(c) Show that two cosets g_1H, g_2H either coincide (if $g_1 = g_2h$ for some $h \in H$) or do not intersect at all.