

MATH 10
ASSIGNMENT 13: OPEN AND CLOSED SETS

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Definition 1. A metric space is a set X with a distance function: for any $x, y \in X$ we have a real number $d(x, y)$ such that

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ for any x, y
3. $d(x, y) = 0$ if and only if $x = y$
4. Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

Usual examples are $\mathbb{R}, \mathbb{R}^2, \dots$, with the Euclidean metric $d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})} = \sqrt{\sum (x_i - y_i)^2}$, but there are other examples as well (problem 1).

Given a point $x \in X$ and a positive real number ε , we define ε -neighborhood of x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

(problem 2).

If $S \subset X$, denote by S' the complement of S . Then, for any $x \in X$, we can have one of three possibilities:

1. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S (in particular, this implies that $x \in S$). Such points are called *interior points* of S ; set of interior points is denoted by $\text{Int}(S)$.
2. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S' (in particular, this implies that $x \in S'$). Thus, $x \in \text{Int}(S')$.
3. Any neighborhood of x contains points from S and points from S' (in this case, we could have $x \in S$ or $x \in S'$). Set of such points is called the *boundary* of S and denoted ∂S .

Note that in particular, the set S itself contains all points of $\text{Int}(S)$, some (possibly none) points of ∂S , and no points from $\text{Int}(S')$.

Definition 2. A set S is called *open* if every point $x \in S$ is an interior point: $S = \text{Int}(S)$.

A set S is called *closed* if $\partial S \subset S$.

HOMEWORK

1. Show that set \mathbb{R}^2 with distance defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

is a metric space. (This distance is sometimes called *Manhattan* or taxicab distance — can you guess why?)

2. The Chebyshev distance in \mathbb{R}^2 is defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Draw the unit disc around the origin $B_1(0)$ using the Euclidean metric, then the Manhattan metric, then the Chebyshev metric in \mathbb{R}^2 .

3. For each of the following subsets of \mathbb{R} , find its interior and boundary and determine if it is open, closed, or neither.
 - (a) Set $\mathbb{N} = \{1, 2, 3, \dots\}$.
 - (b) Interval $[0, 1]$
 - (c) Open interval $(0, 1)$
 - (d) Interval $[0, 1)$.
 - (e) Set of all rational numbers
 - (f) Set consisting of just two points $\{0, 1\}$

*(g) Set $x^3 + 2x + 1 > 0$

Are there any subsets of \mathbb{R} which are both open and closed?

4. Show that a set S is open if and only if its complement S' is closed. [Hint: note that $\partial S = \partial S'$.]

- *5.** For a set S , let $\bar{S} = S \cup \partial S = \{x \mid \text{In any neighborhood of } x, \text{ there are elements of } S\}$.
Prove that \bar{S} is closed. (It is called the closure of S .)
- 6.** Show that union and intersection of two open sets is open. Is it true if we replace two sets by any collection of open sets?
Same question about closed sets.