MATH 10
ASSIGNMENT 12: ROUCHÉ-CAPELLI THEOREM

DEC 20, 2020

Today, we try to summarize what we learned about systems of linear equations, while also reviewing the homeworks. The main result the comes out of our previous lessons is the following theorem, sometimes called the Rouché-Capelli theorem.

Theorem. *A system of linear equations has solutions if and only if the rank of its matrix of coefficients equals the rank of its augmented matrix. Moreover, if the system has solutions, then the dimension of the space of solutions is given by the difference between the number of variables and the rank of the matrix of coefficients.*

In particular, this says that, if the system of equations has solutions and the rank of the matrix of coefficients is equal to the number of variables, then the solution is unique. Now, we know that in that situation the matrix of coefficients is invertible. These two facts are related. writing the system of linear equations as $A\mathbf{x} = \mathbf{y}$, then we can use the inverse of A to get $\mathbf{x} = A^{-1}\mathbf{y}$, and the solution is indeed unique.

Yet another way of understanding this situation is in terms of linear maps. In that case the matrix equation $A\mathbf{x} = \mathbf{y}$ represents a linear equation $f(\mathbf{x}) = \mathbf{y}$ in a given basis. In this case the linear map f is invertible, so the solution is simply $\mathbf{x} = f^{-1}(\mathbf{y})$.

Homework (Optional)

1. This problem is a review of many of the ideas we learned. Consider the linear function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\6\\11\end{bmatrix},\ f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\5\\10\end{bmatrix},\ f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\5\end{bmatrix}$$

(a) Compute

$$f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) \text{ and, in particular, } f\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right).$$

(b) Find the image of this function. If the image is a plane (or a line) find the equation(s) of the plane (or line).

(c) For a given vector $\vec{v}$ in the image set of f, find the inverse image of this vector, $f^{-1}(\vec{v})$. If this is a plane (or a line), find the equation(s) of the plane (or line).