MATH 10
ASSIGNMENT 11: LINEAR MAPS
DECEMBER 13, 2020

Two classes ago, we learned the abstract concept of vector spaces. Let us refresh those ideas.

Vector Spaces

A *real vector space* is a set V together with two operations: vector addition

$$V \times V \to V : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} + \mathbf{w}$$

and multiplication by a scalar

$$\mathbb{R} \times V \to V : (a, \mathbf{v}) \mapsto a\mathbf{v},$$

such that these operations satisfy the following properties:

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
\mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} \\
\exists \mathbf{0} \in V : \mathbf{0} + \mathbf{v} &= \mathbf{v}, \forall v \in V \\
\forall \mathbf{v} \in V \exists - \mathbf{v} \in V : \mathbf{v} + (-\mathbf{v}) &= \mathbf{0} \\
a(b\mathbf{v}) &= (ab)\mathbf{v} \\
1\mathbf{v} &= \mathbf{v}, \forall \mathbf{v} \\
a(\mathbf{u} + \mathbf{v}) &= a\mathbf{u} + a\mathbf{v} \\
(a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v}.
\end{aligned}$$

So we can roughly say that a vector space is a set where you know how to add the elements and to multiply them by numbers.

Linear Maps

Now consider two vector spaces V, W. We can look at functions between these two sets, $f : V \to W$. The question that comes up is: can some of these functions have special properties that only appear because V and W are vector spaces? The answer is "yes, f can be a *linear map*".

A function $f : V \to W$ is called a *linear map* if, for any vectors $\mathbf{u}, \mathbf{v} \in V$ and any number $a \in \mathbb{R}$, the following properties hold:

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}), \; f(a\mathbf{u}) = af(\mathbf{u}).$$

We say that a function is a linear map if it respects the vector space operations of addition and product by a number. One can show that linear maps can be combined in a few special ways: they can be summed, multiplied by a number, or composed, and the result is a linear map as well (exercise 1).

Introducing a Basis

We also saw the important concepts of basis and dimension: every vector space V has a fixed dimension d (which we assume finite), which means that is it possible to find d vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$ such that any other vector is a combination of these with some coefficients,

$$\mathbf{x} = \sum_{i=1}^{d} x_i \mathbf{e}_i = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + ... + x_d\mathbf{e}_d$$

The vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$ are called a *basis*. In terms of a basis, the sum of vectors and product by a number become very familiar:

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + ... + x_d\mathbf{e}_d) + (y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + ... + y_d\mathbf{e}_d) \\
&= (x_1 + y_1)\mathbf{e}_1 + (x_2 + y_2)\mathbf{e}_2 + ... + (x_d + y_d)\mathbf{e}_d, \\
\alpha\mathbf{x} &= \alpha(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + ... + x_d\mathbf{e}_d) \\
&= (\alpha x_1)\mathbf{e}_1 + (\alpha x_2)\mathbf{e}_2 + ... + (\alpha x_d)\mathbf{e}_d,
\end{aligned}$$

which means that we can simply work with the "vectors of components",

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix},$$

and the operations of addition of vectors and multiplication then take the usual form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_d + y_d \end{bmatrix},$$

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_d \end{bmatrix}. \qquad \alpha \in \mathbb{R}$$

This gives a correspondence between a d-dimensional vector space and $\mathbb{R}^d$.

Linear Maps and Matrices

What happens when we express a linear map $f : V \to W$ from an n-dimensional vector space V to an m-dimensional vector space W in terms of bases in V and W? Let us denote the two bases as $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ and $\mathbf{l}_1, \mathbf{l}_2, ..., \mathbf{l}_m$ and then decompose both the argument and the image of the function f (we will complete some details in exercise 2):

$$\begin{aligned} \mathbf{y} &= f(\mathbf{x}) \Rightarrow \\ y_1\mathbf{l}_1 + y_2\mathbf{l}_2 + ... + y_m\mathbf{l}_m &= f_1(\mathbf{x})\mathbf{l}_1 + f_2(\mathbf{x})\mathbf{l}_2 + ... + f_m(\mathbf{x})\mathbf{l}_m \\ &= f_1(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + ... + x_n\mathbf{e}_m)\mathbf{l}_1 + ... + f_m(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + ... + x_n\mathbf{e}_m)\mathbf{l}_m \\ &= (f_{11}x_1 + f_{12}x_2 + ... + f_{1n}x_n)\mathbf{l}_1 + ... + (f_{m1}x_1 + f_{m2}x_2 + ... + f_{mn}x_n)\mathbf{l}_m. \end{aligned}$$

If you look at the components two sides of the equation we get

$$\begin{aligned} y_1 &= f_{11}x_1 + f_{12}x_2 + ... + f_{1n}x_n \\ y_2 &= f_{21}x_1 + f_{22}x_2 + ... + f_{2n}x_n \\ &\vdots \\ y_m &= f_{m1}x_1 + f_{m2}x_2 + ... + f_{mn}x_n. \end{aligned}$$

But this is just a matrix equation! Remembering the product of matrices this becomes

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and we say that a linear map $f : V \to W$ from an n-dimensional vector space to an m-dimensional vector space is represented by a matrix of order $[m, n]$. Under this correspondence, sum, product by a scalar and composition become sum, product by a scalar and multiplication of matrices (exercise 3)!

This connection allows us to extend the definition of rank to linear maps.

Definition. Let A be an $[m, n]$ matrix. Then the *rank* of A, *rank*(A) is the number of nonzero rows of the matrix when it is put in row echelon form. Similarly, for a linear map f between vector spaces V and W, $f : V \to W$, the *rank* of f is defined as the *rank* of the matrix corresponding to f in given bases in V and W.

Linear Equations

A *linear equation* is an equality of the form $\vec{y} = f(\vec{x})$, where f is a given linear function and $\vec{y}$ is a given vector. The set of solutions is the set of vectors $\vec{x}$ which satisfy the equation.

A simple case is when we have $f : \mathbb{R} \to \mathbb{R}$. Then the equation can be written as (exercise 2) $y = ax$, where $a, y \in \mathbb{R}$ are given and $x \in \mathbb{R}$ is to be found. We have that, if $y = 0$, then $x = 0$ is a solution. If additionally $a \neq 0$, this is the unique solution. On the other hand, if $a = 0$, then actually any real number x is a solution. In case $y \neq 0$ and $a \neq 0$, there is a unique solution $x = y/a$, while if $y \neq 0$ and $a = 0$ then there is no solution.

What about the general case? Given an equation $f(\mathbf{x}) = \mathbf{y}$, we can write it as a matrix equation $Fx = y$:

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

and consequently as a system of linear equations

$$\begin{aligned} f_{11}x_1 + f_{12}x_2 + ... + f_{1n}x_n &= y_1 \\ f_{21}x_1 + f_{22}x_2 + ... + f_{2n}x_n &= y_2 \\ &\vdots \\ f_{m1}x_1 + f_{m2}x_2 + ... + f_{mn}x_n &= y_m. \end{aligned}$$

In this way, matrices, linear maps and systems of linear equations are all related. In particular, we can express the following theorem (from a few classes ago) in the language of linear maps:

Theorem. *Let $f : V \to W$ be a linear function. Suppose that, for a given $w \in W$, the linear equation $w = f(v)$ has solutions $v \in V$. Then the dimension d of the space of solutions is given by*

$$d = dim(V) - rank(f).$$

Homework

1. Consider linear maps $f : U \to V$, $g : U \to V$ and $h : V \to W$ between vector spaces U, V and W.
   (a) Show that the function $(f + g) : U \to V$ defined by $(f + g)(\mathbf{u}) = f(\mathbf{u}) + g(\mathbf{u})$ is a linear map.
   (b) Show that, for any number $\alpha \in \mathbb{R}$, the function $(\alpha f) : U \to V$ defined by $(\alpha f)(\mathbf{u}) = alphaf(\mathbf{u})$ is a linear map.
   (c) Show that the composition $(h \circ g) : U \to W$ defined by $(h \circ g)(\mathbf{u}) = h(g(\mathbf{u}))$ is a linear map.

2. (a) Show that, if $f : \mathbb{R} \to \mathbb{R}$ is a linear function, then $f(x) = ax$ for some $a \in \mathbb{R}$.
   (b) Consider the equation $\vec{y} = f(\vec{x})$, where f is a linear function $f : V \to W$, V is n-dimensional and W is m-dimensional. Now use bases to write $\mathbf{x} = x_1\mathbf{d}_1 + x_2\mathbf{d}_2 + ... + x_n\mathbf{d}_n$ and $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + ... + y_m\mathbf{e}_m$. Show that each component y_i is a linear function of $(x_1, x_2, ..., x_n)$.
   (c) Use the ideas from parts (a) and (b) to show that there are real numbers $a_{11}, a_{12}, ..., a_{1n}$ such that $y_1(x_1, x_2, ..., x_n) = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n$.
   (d) Generalize part (c) for all the $y_i(x_1, ..., x_n)$ to show that $\mathbf{y} = f(\mathbf{x})$ can be written as a matrix equation.

3. Let F, G and H be the matrices corresponding to the linear maps $f : V \to W$, $g : V \to W$ and $h : W \to Z$ (by choosing some bases)
   (a) Show that the matrix corresponding to the linear map $(f + g) : V \to W$ is $F + G$.
   (b) Show that the matrix corresponding to the linear map $(\alpha f) : V \to W$, where α is some real number, is αF.
   (c) Show that the matrix corresponding to the linear map $(h \circ g) : V \to Z$ is HG.