MATH 10
ASSIGNMENT 10: MATRIX PRODUCT
DECEMBER 6, 2020

Last class we saw that matrices of given order $\mathbb{M}[m,n]$ (m rows and n columns) can be added and also multiplied by a number:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}, \tag{1}$$

$$c * \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{bmatrix}, \tag{2}$$

and that these give the set of matrices of a fixed order the structure of a vector space (which means that these operations satisfy a number of nice properties). Notice that one can only add two matrices which have the same order (same number of rows and same number of columns).

Matrix Product

It turns out that we can also define a product of two matrices.

Definition. Let A be an $[m,n]$ matrix and B and $[n,p]$ matrix. Then the *matrix product* AB is the $[m,p]$ matrix whose element at the i-th row and j-th column is the dot product of the i-th row of A with the j-th column of B, ie. $(AB)_{ij} = \sum_k a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{in}b_{nj}$.

Here is an example:

$$\begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 5 & 13 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 5\cdot 5 + 3(-3) & 1(-1) + 5\cdot 13 + 3\cdot 7 \\ 2\cdot 0 + 7\cdot 5 + 4(-3) & 2(-1) + 7\cdot 13 + 4\cdot 7 \end{bmatrix} = \begin{bmatrix} 16 & 85 \\ 23 & 117 \end{bmatrix} \tag{3}$$

Note that we can only multiply two matrices if the number of columns of the first one equals the number of rows of the second.

It is natural to ask if there is a neutral element for matrix multiplication (like the number 1 for multiplication of numbers). The answer is yes, and we denote by I_n the *identity matrix* of order n. This is the $[n,n]$ square matrix whose elements $I_{i,j}$ are 1 if $i = j$ and 0 otherwise:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Note that this matrix indeed works like an identity for matrix multiplication: $AI_n = A, \forall A \in \mathbb{M}[m,n]$, $I_nB = B, \forall B \in \mathbb{M}[n,p]$.

Matrix Inverse

The next question we ask is whether there is a notion of multiplicative inverse for matrices.

Definition. Given matrices $A \in \mathbb{M}[m,n]$, $B \in \mathbb{M}[n,m]$, we say that B is an *inverse* of A (or A is an inverse of B) if $AB = I_m$ and $BA = I_n$.

From the definition we immediately see that only square matrices might have an inverse. It turns out that not even all square matrices are invertible. We still need the concept of *rank*:

Definition. Let A be an $[m,n]$ matrix. Then the *rank* of A, *rank*(A) is the number of nonzero rows of the matrix when it is put in row echelon form.

Next week we will prove the full invertibility criterion:

Theorem 1. *Let $A \in \mathbb{M}[m,n]$. Then A has an inverse if and only if $n = m$ and rank$(A) = m$. Moreover, if an inverse exists, it is unique.*

Thus we can unambiguously use the notation A^{-1} for *the* inverse of a matrix. There are many methods to compute the inverse of a matrix. One you can use is the following. For a given invertible $A \in \mathbb{M}[n,n]$, write the augmented matrix $A|I_n$. By elementary row operations (the operations you would use to bring a matrix to row echelon form), bring the augmented matrix to the form $I_n|B$. Then $B = A^{-1}$.

Homework

1. Consider the following matrices

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & -5 & 0 \\ 3 & 2 & 10 \end{bmatrix}, C = \begin{bmatrix} 4 & 1 \\ 7 & -2 \\ 0 & 2 \end{bmatrix}.$$

Calculate AB, BC and CA.

2. Determine if each one of these matrices is invertible. If yes, find the inverse. Then do the products AA^{-1} and $A^{-1}A$ to check that you get the identity matrix.

(a)

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{bmatrix}.$$

3. Prove that $AI_n = A$ for any $[m,n]$ matrix A and that $I_nB = B$ for any $[n,p]$ matrix B.

4. Prove the following properties of matrix operations:

(a) $A(B+C) = AB + AC$ (distributive property)

(b) $A(BC) = (AB)C$ (associative property)

5. Is it true that $AB = BA$ (commutative property)? If yes, prove it. If not, give a counterexample (that is, find two matrices A and B such that $AB \neq BA$).