MATH 10
ASSIGNMENT 9: VECTOR SPACES AND DIMENSION
NOVEMBER 22, 2020

Today we will introduce some concepts from the general theory of vector spaces. However, we will not be very rigorous, but will rather focus on understanding how these concepts apply to the examples we have already encountered.

Vector Spaces

A *real vector space* is a set V together with two operations: vector addition

$$V \times V \to V : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} + \mathbf{w} \tag{1}$$

and multiplication by a scalar

$$\mathbb{R} \times V \to V : (a, \mathbf{v}) \mapsto a\mathbf{v}, \tag{2}$$

such that these operations satisfy the following properties hold:

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
\mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} \\
\exists \mathbf{0} \in V : \mathbf{0} + \mathbf{v} &= \mathbf{v}, \forall v \in V \\
\forall \mathbf{v} \in V \exists - \mathbf{v} \in V : \mathbf{v} + (-\mathbf{v}) &= \mathbf{0} \\
a(b\mathbf{v}) &= (ab)\mathbf{v} \\
1\mathbf{v} &= \mathbf{v}, \forall \mathbf{v} \\
a(\mathbf{u} + \mathbf{v}) &= a\mathbf{u} + a\mathbf{v} \\
(a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v}.
\end{aligned}$$

Examples

I. The standard example is geometric vectors (arrows) in the plane or in Euclidean space. Addition is given by the parallelogram rule, and multiplication by a scalar by changing the length of the vector (and changing the direction if the number is negative).

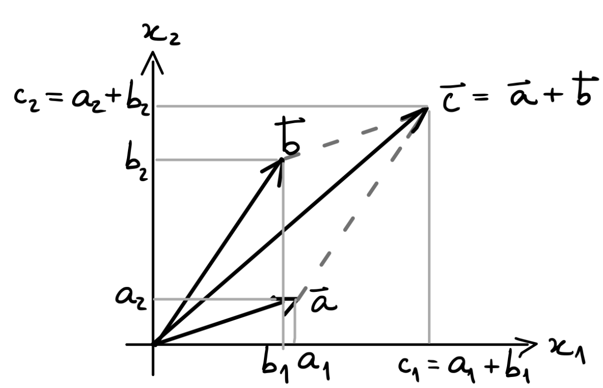

Figure 1. Vector addition in coordinates.

0.1. **II.** $\mathbb{R}^n$, the set of n-tuples of real numbers $x_1, x_2, \ldots, x_n$. We will write them as a column of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

And the operations of addition of vectors and multiplication by numbers are

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \qquad c \in \mathbb{R}$$

III. $\mathbb{M}[m, n]$, the space of matrices of order $[m, n]$, is a generalization of the above concept: an array with m lines and n columns of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

Sum of matrices of the same order and multiplication by a number are defined in analogy to the operations on $\mathbb{R}^n$ defined above.

IV. Polynomials in one variable of degree n (see problem).

By extending the ideas from the previous section, we will also refer to points of $\mathbb{R}^n$ as vectors (starting at the origin and ending at this point).

Basis Vectors

We commonly use coordinates to work with vectors. This is based on the fact that, given any two fixed noncolinear vectors $\vec{e}_1$, $\vec{e}_2$ (called *basis vectors*), we can write all vectors $\vec{a}$ in the plane as $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$ for some components (a_1, a_2). In terms of the components (called *coordinates*), we have

$$\vec{a} + \vec{b} = (a_1\vec{e}_1 + a_2\vec{e}_2) + (b_1\vec{e}_1 + b_2\vec{e}_2) = (a_1 + b_1)\vec{e}_1 + (a_2 + b_2)\vec{e}_2 \tag{3}$$

(see fig.1) and

$$c\vec{a} = c(a_1\vec{e}_1 + a_2\vec{e}_2) = (ca_1)\vec{e}_1 + (ca_2)\vec{e}_2. \tag{4}$$

Thus this gives a correspondence between vectors in the plane with addition and multiplication by a number as defined in **I**, and ordered pairs $(x_1, x_2) \in \mathbb{R}^2$ with the operations of addition and multiplication by numbers as defined in **II**. The concept that relates the two is that of *basis vectors.* The same goes for 3-dimensional vectors, which corresponds to $\mathbb{R}^3$.

Dimension

The number of vectors necessary and sufficient to form a basis is a property of the vector space, called the *dimension.* From the previous point we see that vectors in the plane form a *2-dimensional* vector space, and vectors in the euclidean space form a *3-dimensional* vector space. By using a basis, we can always identify a vector space of dimension d with $\mathbb{R}^d$ just as we did above.

The idea of dimension is very important in solving systems of linear equations. If a system of linear equations has solutions and has d free variables, it means that a general solution depends on the choice of d

numbers $t_1, \dots, t_d$ — the values of free variables. Moreover, in this case the general solution can be written in the form

$$\mathbf{x} = \mathbf{a} + t_1\mathbf{v}_1 + \dots + t_d\mathbf{v}_d$$

for some vectors $\mathbf{a}, \mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{R}^n$. In this case we say that the set of solutions has *dimension* d. It can be shown that the dimension does not depend on how we brought the matrix to row echelon form.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned}$$

the general solution is

$$\mathbf{x} = \begin{bmatrix} -1 + 2t \\ 6 - 3t \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

so the space of solutions has dimension 1.

Theorem. *If a system of linear equations has solutions, then the dimension of the space of solutions is given by*

$$d = (\textit{number of variables}) - (\textit{number of nonzero rows in row echelon form})$$

Indeed, the number of free variables is the number of all variables minus the number of pivot variables.

Thus, typically we expect that a system with n variables and k equations has $n - k$ dimensional space of solutions. This is not always true: it could happen that after bringing it to row echelon form, some rows become zero, or that the system has no solutions at all — but these situations are unusual (at least if $k \leq n$).

Homework

1. Prove that the operations of addition and multiplication by a scalar in the vector space $\mathbb{R}^n$ (example **II**) satisfy all the properties outlined below equation (2).
2. The polynomials of degree n in one variable x form a vector space (remember problem 1 of Homework 8). Can you find a basis? What is the dimension?
3. Consider the space of matrices with 2 lines and 3 columns $\mathbb{M}[2, 3]$. Can you find a basis for this space? What is the dimension? What about the general case $\mathbb{M}[m, n]$?
4. Consider a basis $\vec{e}_x$, $\vec{e}_y$ for the vectors in the plane such that $\vec{e}_x \cdot \vec{e}_x = 1$, $\vec{e}_y \cdot \vec{e}_y = 1$ and $\vec{e}_x \cdot \vec{e}_y = 0$. Now, given two vectors $\vec{v}_1 = x_1\vec{e}_x + y_1\vec{e}_y$ and $\vec{v}_2 = x_2\vec{e}_x + y_2\vec{e}_y$, express $\vec{v}_1 \cdot \vec{v}_2$ in terms of their components. (Note, this extends the correspondence between geometric vectors in the plane and $\mathbb{R}^2$ to the dot product.)