PERMUTATIONS

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A **permutation** of some set S is a way of reordering, or permuting, elements of S. Mathematically, it can be described as a function $f\colon S \to S$ which is a bijection (one-to-one and onto, or invertible). We will only be discussing permutations of finite sets, usually the set $S = \{1, \dots, n\}$. In this case one can also think of a permutation as a way of permuting n items placed in boxes labeled $1, \dots, n$: namely, move item from box 1 to box $f(1)$, item from box 2 to $f(2)$, etc. The set of all permutations of $\{1, \dots, n\}$ is denoted by S_n.

Permutations can be composed in the usual way: $f \circ g(x) = f(g(x))$.

Notation: the permutation f which sends 1 to a_1, 2 to a_2, etc, is usually written as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

An alternative way of writing permutations is using cycles. A **cycle** $(a_1 a_2 \dots a_k)$ is a permutation which sends a_1 to a_2, a_2 to a_3, …, a_n to a_1 (and leaves all other elements unchanged). For example, (123) is the permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(a) = a$ for all other a. The same cycle can also be written as (231).

We can also consider products (i.e. compositions) of several cycles. For example, (123)(45) is a permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f(4) = 5$, $f(5) = 4$. It is also customary not to write cycles of length one: instead of writing (123)(4), we write just (123).

1. Fifteen students are meeting in a classroom which has 15 chairs numbered 1 through 15. The teacher requires that every minute they change seats following this rule:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	5	10	8	11	14	15	6	13	1	4	9	7	2	12

(e.g., the student who was sitting in the chair number 1 would move to chair number 3). In how many minutes will the students return to their original seats?

2. (a) How many permutations of a set of 100 elements there are which are a single cycle of length 100?

(b) How many permutations of a set of 100 elements are there which consist of a cycle of length 50, a cycle of length 23, a cycle of length 24, and cycle of length 3?

(c) How many permutations of a set of 100 elements are there that contain a cycle of length 51?

(d) How many permutations of a set of 100 elements are there that contain a cycle of length more than 50?

3. (This is a famous problem, suggested in 2003 by a Danish computer scientist Peter Bro Miltersen. It is a hard problem, but the previous problem gives a hint.)

The prison warden offers 100 death row prisoners, who are numbered from 1 to 100, a last chance. A room contains a cupboard with 100 drawers. The warden randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategy — but may not communicate once the first prisoner enters to look in the drawers. The prisoners are not allowed to move numbers from one drawer to another or make any other changes.

What is the prisoners' best strategy?

Note: there is no strategy that guarantees the prisoners win, but there are strategies that offer a chance of survival significantly better than $(1/2)^{100}$.