Basic rules

The number of ways to choose k items out of n if the order in which they are chosen matters is
$$_nP_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}.$$
The number of ways to choose k items out of n if the order in which they are chosen doesn't matter is
$$_nC_k = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

For example, the number of sequences of length n consisting of k zeros and $n-k$ ones is $_nC_k$ (this is equivalent to choosing k positions where we put zeros).

1. How many ways there are to divide 10 students into two teams of 5? (Teams do not have names)

2. How many ways there are for 15 students to take seats in a classroom with 30 chairs?

3. You are given m white balls and n black balls ($m > n$). How many way are there to arrange them in a line so that no two black balls are next to each other?

4. You are given an unlimited supply of black and white balls. How many ways are there to create a line of 20 balls so that no two black balls are next to each other? What if you are allowed at most two black balls next to each other?

5. Show that in any collection of 10 different integers, you can select some of them and put signs $+$ and $-$ between them so that the resulting expression is divisible by 1000. Note: you are only allowed to add and subtract, no other operations. [Hint: show that there are two subsets of your collection such that sums of numbers in these subsets are the same modulo 1000.]

6. A committee has 9 members; they store all materials in a safe locked by n locks, and keys are distributed among committee members so that any 6 committee members together can unlock the safe, but no 5 members can do that. Can you find a way of doing this? What is the smallest possible n?

Stars and Bars

8. How many ways there are to arrange 12 books on 2 bookshelves (top and bottom one)? The order on each bookshelf matters; there are no restrictions on how many of the 12 books are on top shelf.

9. How many different monomials in 3 variables x, y, z of total degree n are there? in 4 variables?

10. How many different monomials in 3 variables x, y, z of total degree n are there if we additionally require that each variable appears with positive degree (i.e. we look for monomials $x^a y^b z^c$, $a > 0$, $b > 0$, $c > 0$, $a + b + c = n$).

11. How many ways there are to put 15 chairs in 4 rooms if every room must have at least one chair? (Chairs are all identical, chairs inside the room are not ordered.)

12. How many ways there are to put 15 people in 4 rooms if every room must have at least one person? (People are all different, people inside the room are not ordered.)

Overcounting

A simple idea is that if we want to count number of items in some set (e.g., number of ways to...) and we have found a method of counting that counts every item m times, then correct answer is (result of counting)/m.

This is very commonly used in many problems.

10. How many different "words" can be formed by permuting letters of the word "Mississippi"? (by "word", we mean any sequence of letters, not necessarily meaningful").

11. How many ways there are to group $2n$ people into n pairs?

12. Lynne is making tokens for some game. Each token is a small cube, with faces colored in 6 colors (red, green, blue, yellow, black, white), one face of each color. How many different tokens are possible?

Same question for a token which has the shape of regular dodecahedron (12 faces, each a regular pentagon), using 12 colors.

13. How many different necklaces can one make using 1 green, 5 red, and 7 blue beads? using 2 green, 5 red, and 7 blue beads?