

ADVANCED MATH PROBLEM SOLVING CLUB
SOME AIME PROBLEMS

MARCH 21, 2021

PROBLEMS

1. Zou and Chou are practicing their 100-meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is $\frac{2}{3}$ if they won the previous race but only $\frac{1}{3}$ if they lost the previous race. Find the probability that Zou will win exactly 5 of the 6 races.
2. Find the number of positive integers less than 1000 that can be expressed as the difference of two integral powers of 2.
3. Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.
4. Let $A_1A_2A_3\dots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8 , and A_{12} . At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. Find the expected number of minutes until the frogs stop jumping.
5. Two card players and a spectator are sitting around a table. A card deck is shuffled, and 7 cards are removed from it and shown to everyone. After that, these 7 cards are distributed among them in secret: 1 card to the spectator, 3 to each of the players.

The players can talk to each other, but the spectator will hear everything they say. Can they somehow exchange information so that at the end, each of the players will know what cards the other player has but the spectator doesn't know where any of the 6 cards (not including his own) is?