INVARIANTS

DEC 6, 2020

An invariant is something that does not change.
A semi-invariant is somethign that only changes in one direction (e.e.g, only decreases).

1. Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by a single number: if the numbers were a, b, we replace them by $a+b$. Can you predict which number will be written on the board at the end?
2. Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by a single number: if the numbers were a, b, we replace them by $a+b+ab$. Can you predict which number will be written on the board at the end?
3. In the alphabet used by the tribe OUO there are only two letters, O and U. Two words in their language are synonyms if one word can be obtained from the other by removing pair letters UO (next to each other) or adding anywhere in the word the combinations "OU" and "UUOO". Are the words OUU and UOO synonyms?
4. There are 16 glasses on the table, one of them upside down. You are allowed to turn over any 4 glasses at a time. Can you get all glasses standing correctly by repeating this operation?
5. In the country of RGB, there are 13 red, 15 green and 17 blue chameleons. Whenever two chameleons of different colors meet, both of them change their color to the 3rd one (e.g., if red and green meet, they both turn blue). Do you think it can happen that after some time, all chameleons become the same color?[Hint: give each color a numeric value, say 0, 1, 2]
6. (a) We are given a 4×4 table, each cell containing either + sign or sign:

$$\begin{array}{cccc} + & - & + & + \\ + & + & + & + \\ + & + & + & + \\ + & - & - & + \end{array}$$

   You can reverse all signs in a single row or column, replacing each + by - and - by +. Is it possible to make all signs + by repeating this operation?

   *(b) Same question, but for this table:

$$\begin{array}{cccc} + & - & + & + \\ + & + & + & + \\ + & + & + & + \\ + & - & + & + \end{array}$$

7. (From HMMT Nov 2016, Theme round)

   We have 10 points on a line $A_1, A_2, \ldots A_{10}$ in that order. Initially there are n chips on point A_1. Now we are allowed to perform two types of moves. Take two chips on A_i, remove them and place one chip on A_{i+1}, **or** take two chips on A_{i+1}, remove them, and place a chip on A_{i+2} and A_i. Find the minimum possible value of n such that it is possible to get a chip on A_{10} through a sequence of moves.

   [Hint: except when $i=1$, it is always better to do move (2) instead of move (1).]
8. We have an infinite sheet of square ruled paper (think of it as first quadrant on the coordinate plane), with cells indexed by pairs of positive integers. In the beginning, we have a chip on square $(1,1)$. At every moment, we can make the following move: if there is a chip at square (i,j), and squares above and to the right of it (that is, squares $(i+1,j)$ and $(i,j+1)$) are both empty, we can remove the chip from (i,j) and put a chip in each of the squares $(i,j+1)$ and $(i+1,j)$.

   Using these moves, can we clear the 3×3 square in the corner?
9. A 100×100 yard field of wheat is divided into 1 yd $\times$ 1 yd squares. Initially, 9 of these squares were infected by some crop disease. The disease spreads as follows: for every square, if in the given year at least 2 of its 4 neighbors were infected, then next year the infection spreads to this square. (The squares that were infected stay infected forever). Prove that the disease will never spread to the whole field.

10. Conway's soldiers. This game is a variant of peg solitaire. It takes place on an infinite checkerboard. The board is divided by a horizontal line that extends indefinitely. Above the line are empty cells and below the line are an arbitrary number of game pieces, or "soldiers". As in peg solitaire, a move consists of one soldier jumping over an adjacent soldier into an empty cell, vertically or horizontally (but not diagonally), and removing the soldier which was jumped over. The goal of the puzzle is to place a soldier as far above the horizontal line as possible.

(a) Give a examples of how one can reach rows 1, 2, 3.

(b) Can you reach row 4?

*(c) Prove that it is impossible reach row 5.

The last part is rather hard; to do it, you might want to do a one-dimensional version first — see next problem.

11. Consider one-dimensional analog of Conway's soldiers, in which you have one line of cells indexed by integers. All cells with index $i \geq 0$ are empty, and some of the cells with negative indices contain "soldiers". As before, a move consists of one soldier jumping over an adjacent soldier into an empty cell and removing the soldier which was jumped over.

(a) Show that one can assign "weights" to different cells so that when a soldier jumps to the right, the sum of all weights of cells taken by soldiers does not change, and when one jumps to the left, the sum decreases.

(b) What is the highest position that can be reached in this game?

Hint: look for weights forming geometric progression: weight of cell with index k given by a^k for some suitably chosen real number a.