MATH CLUB: RECURRENT SEQUENCES

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In many problems, a sequence is defined using a recurrence relation, i.e. the next term is defined using the previous terms. By far the most famous of these is the Fibonacci sequence:

$$F_0 = 0, F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1} \tag{1}$$

The first several terms of this sequence are below:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \dots$$

For such sequences there is a method of finding a general formula for nth term, outlined in problem 6 below.

1. Into how many regions do n lines divide the plane? It is assumed that no two lines are parallel, and no three lines intersect at a point.
Hint: denote this number by R_n and try to get a recurrent formula for R_n: what is the relation betwen R_n and R_{n-1}?

2. The following problem is due to Leonardo of Pisa, later known as Fibonacci; it was introduced in his 1202 book *Liber Abaci.*
Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on.
How many pairs will there be in one year?

3. Daniel is coming up the staircase of 20 steps. He can either go one step at a time, or skip a step, moving two steps at a time.
In how many ways can he come up the stairs?
["Way" refers to (ordered) sequences of his moves, e.g. 1, 2, 1,1, 2, 2,... ; each number represents by how many steps he moved, and the sum must be equal to 20.]
Hint: again, write a recurrence formula!

4. How many ways are there to write a 10-letter "word" consisting of letters A and B if we do not allow letter B to appear two times in a row? What if we allow for B at most two times in a row?

5. Prove that the Fibonacci numbers satisfy the following identity: $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$
Can you guess a formula for the sum $F_1^2 + F_2^2 + \cdots + F_n^2$?

6. In this problem, we show how one can derive a formula for Fibonacci number.
Let us call a sequence a_n a *generalized Fibonacci sequence* (GFC) if it satisfies the same recurrence relation ($a_{n+1} = a_n + a_{n-1}$), but might have different first two terms.

(a) Show that a geometric progression $a_n = \lambda^n$, $\lambda \neq 0$, is a GFC if and only if λ satisfies the equation

$$\lambda^2 = \lambda + 1 \tag{2}$$

Find the roots of this equation.

(b) Let λ_1, λ_2 be the two roots of equation (2). Show that then any sequence of the form

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n \tag{3}$$

(where c_1, c_2 are some constants that do not depend on n) is a GFC.

(c) Find constants c_1, c_2 so that the sequence a_n defined by (3) satisfies $a_0 = 0$, $a_1 = 1$.

(d) Write a general formula for F_n.

7. Use a calculator to estimate how large F_{1000} is.

8. Pell numbers are defined by the relations

$$P_0 = 0, P_1 = 1, \qquad P_{n+1} = 2P_n + P_{n-1} \tag{4}$$

Compute several Pell's numbers by hand; then try to modify the method of problem 6 to get a formula for Pell's numbers.

9. Show that for large n, the ratio $(P_{n-1} + P_n)/P_n$ is close to $\sqrt{2}$. Write the approximation one gets in this way for $n = 8$ and check how close it is to the actual value. (This series of approximations to $\sqrt{2}$ was known already in 4th century BC).

10. (This problem suggested by Alexander Kirillov Sr; it is a closely related to problem 1).

In 1950s, there were seven high rise buildings in the city of Moscow (one of them was Moscow State University). You could see them from any point in the city. Of course, the order in which these buildings appear on the skyline would depend on the observation point. [here the order means a cyclic order, that is, a way of arranging objects on a circle]. Is it true that all possible cyclic orders of these buildings could be observed from some point in the in the city?

You can assume that no three of these buildings were on the same line.