ASSIGNMENT 1: NUMBER THEORY AND MORE!

SEPTEMBER 27, 2020

Simple problems

1. Find the remainder upon the division of 17^{2020} by 7.

2. In how many zeroes does the number 100! end?

3. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7

4. How many perfect squares are divisors of of the product $1! \cdot 2! \cdots 9!$?

5. Prove that, given any prime $p > 5$, there is a number of the form $111 \ldots 1$ which is divisible by p. [Hint: can you first show that there are two numbers of this form that have the same remainder upon division by p]

Harder problems

1. (a) Write the number 1/7 as an infinite decimal. Can you prove that the digits of this number will be repeating periodically?[Hint: do not use calculators; work by hand, recalling how long division works.]

(b) Write the numbers 2/7, 3/7, ... 6/7 as decimals. DO you see any interesting coincidences? can you try and explain why?

2. You have two identical glass balls. You want to determine what is the maximal height from which such a ball can be dropped without breaking, by dropping it from windows at different floors of a 100-story building.

How many attempts will you need? clearly, you can find the answer in 100 attempts - can you do better than that?

3. Moscow subway has a size restriction for packages: **sum** of the three dimensions (width, length, height) must be at most 120cm. Is it possible to cheat the system by taking a package with sum of dimensions more that 120 and fitting it inside a box with sum of dimensions 120?

[This is a very hard problem. First of all, you might want to consider the 2-dimensional case first. A further hint is on reverse.]

Hint for last problem: in 2-dimensional case, given a rectangle R, cosider its d-neighborhood: all points at distance at most d from R (including points of R itself). Can you find a formula for $S_d(R)$, the area of such a d-neighborhood, in terms of the width and length of the triangle?