Classwork 27.

Can we complete square in generally and find the general form of the root of quadratic equation?

Polynomial of the second order can be written generally as:

$$ax^2 + bx + c$$

a, b, c are real numbers, $a \neq 0$ and x is a variable.

First, let's move the common factor (fist coefficient) out:

$$ax^2 + bx + c = a\left(\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Do you remember the algebraic identity

$$(k+z)^2 = (k+z)(k+z) = k^2 + 2kz + z^2$$

The last expression is

$$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Looks like $k^2 + 2kz + z^2$.

$$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = \left(x^2 + 2 \cdot \frac{1}{2}\frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right)$$

$$= \left(x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right)$$

We can further transform the expression:

$$\left(x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right) = \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right)^2 =$$

Last part is actually another identity:

$$p^2 - q^2 = (p+q)(p-q):$$

Therefore, we can rewrite it as

$$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right)^2 = \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}\right)^2 = \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2$$

$$= \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

Polynomial of the second order can be factorized as

$$ax^2 + bx + c = a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

a, b, c are numbers. Do you think any quadratic polynomial can be factorized?

Can we use this factorization process to find the way to solve a quadratic equation? Quadratic polynomial is an expression in the form

$P(x) = ax^2 + bx + c$

$P(x)$ means that the value of the expression depends of the value of x. What if we want to find such x that the polynomial $P(x)$ will have a certain value? For example, which x will bring the polynomial (I will omit the word "quadratic" now, but we remember, that we are talking about the quadratic polynomials) $x^2 - 6x + 13$ to 5. In other words, can we solve the equation:

$$x^2 - 6x + 13 = 5$$

First, let's rewrite this equation as

$$x^2 - 6x + 13 - 5 = 0$$
$$x^2 - 6x + 8 = 0$$

Such equations we are going to call the quadratic equation.
We have just proved that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

For any $a, b,$ and c, $a \neq 0$. If $a = 0$, the polynomial $P(x)$ is not quadratic.

$$a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

Can be equal to 0 only if either of the expressions inside of the parenthesis is equal to 0.
If

$$x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0; \quad x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$$

$$x_1 = -\left(\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = -\left(\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We got two very similar expressions for our solutions:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the discriminant, and denoted as D.

$$D = b^2 - 4ac$$

For each quadratic polynomial we can find $D = b^2 - 4ac$. D is under the radical sign ($\sqrt{\ }$) in the expressions for the roots of quadratic equation, so D can't be less than 0.

1. If $D > 0$, the equation has different roots
2. If $D = 0$, the second part of the expressions for x becomes 0, and both roots are the same, $-\frac{b}{2a}$, and polynomial is a full square,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

3. If $D < 0$, then the square root of a negative number can't be a real number and the equation doesn't have any solution among the real numbers (but has solutions in a set of complex numbers; complex numbers are outside of the scope of our class, you will learn about them later.)

We can also rewrite the expression for factorization of the quadratic polynomial using the discriminant:

$$P(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$
$$= a\left(x + \frac{b - \sqrt{D}}{2a}\right)\left(x + \frac{b + \sqrt{D}}{2a}\right)$$

1. Solve the following quadratic equations:

$a.\ x^2 - 6x + 8 = 0$ $\qquad$ $b.\ x^2 - x - 2 = 0$

$c.\ x^2 + 4x + 15 = 0$ $\qquad$ $d.\ 5x^2 + 8x - 9 = 0$

$e.\ 3x^2 - 5x - 2 = 0$ $\qquad$ $b.\ x^2 + 5x + 6 = 0$

Vieta's formula for the root of the quadratic equations.

We derived the general formula for the root of quadratic equation.

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{\left(-b + \sqrt{b^2 - 4ac}\right) \cdot \left(-b - \sqrt{b^2 - 4ac}\right)}{4a^2}$$
$$= \frac{b^2 + b\sqrt{b^2 - 4ac} - b\sqrt{b^2 - 4ac} - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}$$

$$x_1 + x_2 = -\frac{b}{a}$$
$$x_1 \cdot x_2 = \frac{c}{a}$$

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2. Two pipes fill together a pool in 1 h and 20 minutes. If the fist pipe is open for 10 minutes, and the second pipe is open for 12 minutes, the pool will be filled on 2/15. How fast each pipe will fill the pool?
3. Simplify the expression:

$$(x^2 + y^2 + x + y)(x + y + xy) =$$

4. In a restaurant, customer can order a cheese platter for $15 or $20. For $15 platter, you can choose 3 different kind of cheese out of 15 and for $20 platter you can choose 5 different kind of cheese. How many different ways are there to create these two cheese platters?

5. Using the algebraic identities calculate:

   $a.\ \ 91 \cdot 89;$ $\qquad$ $b.\ \ 61^2;$

6. Solve the equations:

   $a.\ \ |2x - 8| = 10;$

   $b.\ \ \big||x| - 5\big| = 1$

7. The city currently has 48,400 inhabitants. It is known that the population of this city is increasing annually by 10%. How many inhabitants were in the city two years ago?

8.