Classwork 26.

Review:

1. Solve the following:

$$(3x+2)(2-x)(x+28)=0$$

This equation is actually a cubic equation, but the polynomial was already factorized and now we can say, that the equation is equal to 0 if any of the factors is 0. So,

$$x=-\frac{2}{3};\quad x=2,\quad x=-28$$

Factorization of the polynomials not always easy, but for the quadratic polynomials we can try to complete the full square.

2. By completing the square solve the equation:

$$4x^2-7x+3=0$$

We need to factorize the polynomial $4x^2-7x+3$:

$$4x^2-7x+3=4\left(x^2-\frac{7}{4}x+\frac{3}{4}\right)$$

$$x^2-\frac{7}{4}x+\frac{3}{4}=x^2-2\cdot\frac{1}{2}\cdot\frac{7}{4}x+\frac{3}{4}=x^2-2\cdot\frac{7}{8}x+\frac{49}{64}-\frac{49}{64}+\frac{3}{4}=$$

$$\left(x-\frac{7}{8}\right)^2-\frac{49}{64}+\frac{3}{4}=\left(x-\frac{7}{8}\right)^2-\frac{49}{64}+\frac{48}{64}=\left(x-\frac{7}{8}\right)^2-\left(\frac{1}{8}\right)^2=\left(x-\frac{7}{8}-\frac{1}{8}\right)\left(x-\frac{7}{8}+\frac{1}{8}\right)$$

$$=(x-1)\left(x-\frac{3}{4}\right);\quad x=1, x=\frac{3}{4}$$

Let's factorize the polynomial directly:

$$4x^2-7x+3=4x^2-4x-3x+3=4x(x-1)-3(x-1)=(x-1)(4x-3)=0;$$

$$x=1,\qquad x=\frac{3}{4}$$

Can we complete square in generally and find the general form of the root of quadratic equation?

Polynomial of the second order can be written generally as:

$$ax^2+bx+c$$

a, b, c are real numbers, $a\neq 0$ and x is a variable.

First, let's move the common factor (fist coefficient) out:

$$ax^2+bx+c=a\left(\frac{a}{a}x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

Do you remember the algebraic identity

$$(k+z)^2=(k+z)(k+z)=k^2+2kz+z^2$$

The last expression is

$$\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

Looks like $k^2+2kz+z^2$.

$$\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=\left(x^2+2\cdot\frac{1}{2}\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$
$$=\left(x^2+2\cdot\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2\right)-\left(\left(\frac{b}{2a}\right)^2-\frac{c}{a}\right)$$

We can further transform the expression:

$$\left(x^2+2\cdot\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2\right)-\left(\left(\frac{b}{2a}\right)^2-\frac{c}{a}\right)=\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}\right)^2=$$

Last part is actually another identity:

$$p^2-q^2=(p+q)(p-q):$$

Therefore, we can rewrite it as

$$\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}\right)^2=\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2}{4a^2}-\frac{4ac}{4a^2}}\right)^2=\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2$$
$$=\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)$$

Polynomial of the second order can be factorized as

$$ax^2+bx+c=a\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)$$

a, b, c are numbers. Do you think any quadratic polynomial can be factorized?

Can we use this factorization process to find the way to solve a quadratic equation? Quadratic polynomial is an expression in the form

$P(x)=ax^2+bx+c$

$P(x)$ means that the value of the expression depends of the value of x. What if we want to find such x that the polynomial $P(x)$ will have a certain value? For example, which x will bring the polynomial (I will omit the word "quadratic" now, but we remember, that we are talking about the quadratic polynomials) $x^2-6x+13$ to 5. In other words, can we solve the equation:

$$x^2-6x+13=5$$

First, let's rewrite this equation as

$$x^2-6x+13-5=0$$
$$x^2-6x+8=0$$

Such equations we are going to call the quadratic equation.

We have just proved that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

For any $a, b,$ and c, $a \neq 0$. If $a = 0$, the polynomial $P(x)$ is not quadratic.

$$a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

Can be equal to 0 only if either of the expressions inside of the parenthesis is equal to 0.
If

$$x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0; \quad x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$$

$$x_1 = -\left(\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = -\left(\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We got two very similar expressions for our solutions:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the discriminant, and denoted as D.

$$D = b^2 - 4ac$$

For each quadratic polynomial we can find $D = b^2 - 4ac$. D is under the radical sign ($\sqrt{}$) in the expressions for the roots of quadratic equation, so D can't be less than 0.

1. If $D > 0$, the equation has different roots
2. If $D = 0$, the second part of the expressions for x becomes 0, and both roots are the same, $-\frac{b}{2a}$, and polynomial is a full square,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

3. If $D < 0$, then the square root of a negative number can't be a real number and the equation doesn't have any solution among the real numbers (but has solutions in a set of complex numbers; complex numbers are outside of the scope of our class, you will learn about them later.)

We can also rewrite the expression for factorization of the quadratic polynomial using the discriminant:

$$P(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$
$$= a\left(x + \frac{b - \sqrt{D}}{2a}\right)\left(x + \frac{b + \sqrt{D}}{2a}\right)$$