MATH 8: HANDOUT 21
DIVISIBILITY IV: PYTHAGOREAN TRIPLES

Pythagorean triples

In this handout we will take a look at Pythagorean triples, i.e. triples of natural numbers such that

$$x^2 + y^2 = z^2 \tag{1}$$

Of course, we are all familiar with a triple $(3, 4, 5)$, and maybe with $(5, 12, 13)$. But what are the others?

We will describe how to find all of those triples, using two approaches: arithmetic and geometric.

Arithmetic Approach

1. Notice that if (x, y, z) is a Pythagorean triple, then so is (kx, ky, kz) for any $k \in \mathbb{N}$. That is, $(3, 4, 5)$ is a Pythagorean triple, and so are $(6, 8, 10)$, $(15, 20, 25)$, etc. In what follows, we will limit ourselves only to the triples where all numbers are relatively prime, i.e. they don't have a common divisor.

2. Now notice, that if three numbers in a Pythagorean triple (x, y, z) are relatively prime, then each pair of numbers is also relatively prime. [**Note:** This is not true in general – numbers $(2, 4, 7)$ are relatively prime as a triple, but $(2, 4)$ are not!]

Proof. Let's do a proof by contradiction. Assume that x and y are not relatively prime, then there is a common divisor d such that $x = dk$ and $y = dl$. Then

$$x^2 + y^2 = d^2k^2 + d^2l^2 = d^2(k^2 + l^2) = z^2$$

This means that z^2 is divisible by d^2, and therefore z is divisible by d. Therefore, all (x, y, z) are divisible by d and therefore they are not relatively prime. Contradiction. $\square$

3. This relative "primeness" means that all numbers (x, y, z) are completely different in their prime factorizations: all prime factors of x, y, z are different.

4. Now, that means that:
   (a) among (x, y, z) there can't be more than one even number; and
   (b) all numbers can't be odd as well, since odd $\pm$ odd $=$ even.

5. z is odd.

Proof. Consider remainders upon division by 4. Squares always have remainder 0 or 1:

$$\begin{array}{lll} \text{If} & n = 2k: & n^2 = 4k^2, r = 0 \\ \text{If} & n = 2k+1: & n^2 = (2k+1)^2 = 4k^2 + 4k + 1, r = 1. \end{array}$$

If z is even, then x and y are odd; the remainder of $x^2 + y^2$ upon division by 4 is therefore 2, but the remainder of z^2 upon division by 4 is 0 — contradiction with the fact that $x^2 + y^2 = z^2$. Therefore, z is odd. $\square$

6. Since z is odd, it means that either x or y is even. Let's assume that x is even: $x = 2t$. Then

$$\begin{aligned} x^2 + y^2 &= z^2 \\ 4t^2 + y^2 &= z^2 \\ 4t^2 &= z^2 - y^2 \\ 4t^2 &= (z-y)(z+y) \\ t^2 &= \frac{z-y}{2} \cdot \frac{z+y}{2}, \end{aligned}$$

and all numbers in the above equality are whole.

7. Lemma:

$$\gcd\left(\frac{z-y}{2}, \frac{z+y}{2}\right) = 1.$$

Proof. Let p be a common divisor of $\frac{z-y}{2}$ and $\frac{z+y}{2}$. Then:

$$\begin{array}{lll} p & \text{is a factor of} & \dfrac{z+y}{2} - \dfrac{z-y}{2} = y; \\ p & \text{is a factor of} & \dfrac{z+y}{2} + \dfrac{z-y}{2} = z. \end{array}$$

Therefore, z and y are not relatively prime — contradiction. □

8. Since $\frac{z-y}{2}$ and $\frac{z+y}{2}$ are relatively prime, their prime factorizations have different factors:

$$\begin{aligned} \frac{z-y}{2} &= p_1 p_2 \dots p_i \\ \frac{z+y}{2} &= q_1 q_2 \dots q_j \end{aligned}$$

where all p's are different from all q's. Also we know that

$$t^2 = \frac{z-y}{2} \cdot \frac{z+y}{2}.$$

Prime factorization of t^2 has prime factors in even powers — and therefore prime factors in prime factorizations of $\frac{z-y}{2}$ and $\frac{z+y}{2}$ also have even powers. Therefore, it is possible to take square roots of them:

$$\begin{aligned} &\frac{z-y}{2} = n^2; \quad \frac{z+y}{2} = m^2 \\ &t^2 = \frac{z-y}{2} \cdot \frac{z+y}{2} = n^2 m^2 \\ &t = mn \end{aligned}$$

From here we can get the expressions for x, y, and z:

$$\begin{aligned} &x = 2t = 2mn; \\ &y = \frac{z+y}{2} - \frac{z-y}{2} = m^2 - n^2; \\ &z = \frac{z+y}{2} + \frac{z-y}{2} = m^2 + n^2. \end{aligned}$$

Any Pythagorean triple can be obtained this way: we choose any natural m and n, and then calculate x, y, and z.

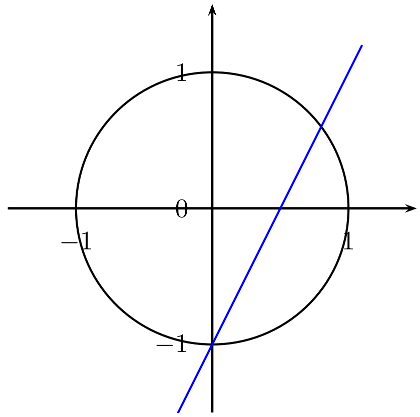
Example: If $m = 5, n = 3$ then

$$\begin{aligned} &x = 2mn = 2 \cdot 5 \cdot 3 = 30 \\ &y = m^2 - n^2 = 5^2 - 3^2 = 25 - 9 = 16 \\ &z = m^2 + n^2 = 5^2 + 3^2 = 25 + 9 = 34 \end{aligned}$$

We can of course easily verify that $16^2 + 30^2 = 34^2$.

Geometric Approach

1. Let us divide the equality $a^2 + b^2 = c^2$ by c^2. Then, if $x = a/c$ and $y = b/c$, $x^2 + y^2 = 1$. That is, point (x, y) has rational coordinates (since a, b, c are whole numbers) and belongs to the unit circle.
2. To find such points we will be considering lines that go through point $(0, -1)$ with rational slope. Since the line goes through the point $(0, -1)$, i.e. y-intercept is -1, then such lines have equation $y = kx - 1$.



3. Now notice, if the point is rational, i.e. $(x, y) = (4/5, 3/5)$, then the slope of the line will also be rational: in our example, $y = 2x - 1$:

$$\frac{3}{5} = k \cdot \frac{4}{5} - 1$$
$$k \cdot \frac{4}{5} = \frac{8}{5}$$
$$k = 2$$

Similarly, you can verify that for Pythagorean triple corresponding to $(12/13, 5/13)$, the equation fo the line is $y = 1.5x - 1$.

4. Now, we will prove that the converse is also true: for any rational k, the line $y = kx - 1$ intersects the unit circle in a rational point.

Proof. We need to find intersection of a circle $x^2 + y^2 = 1$ and a line $y = kx - 1$. Let us plug in the expression for y into the equation of the circle:

$$x^2 + y^2 = 1$$
$$x^2 + (kx - 1)^2 = x^2 + k^2x^2 - 2kx + 1 = 1$$
$$x^2(k^2 + 1) = 2kx$$

From here we get either $x = 0$ or

$$x = \frac{2k}{k^2 + 1}$$
$$y = kx - 1 = k\frac{2k}{k^2 + 1} - 1 = \frac{2k^2}{k^2 + 1} - 1 = \frac{2k^2 - k^2 - 1}{k^2 + 1} = \frac{k^2 - 1}{k^2 + 1}$$

and both of these expressions are rational. □

5. Now, since k is a rational number, set $k = m/n$. Then we get:

$$x = \frac{2\frac{m}{n}}{(\frac{m}{n})^2 + 1} = \frac{2mn}{m^2 + n^2}$$
$$y = \frac{k^2 - 1}{k^2 + 1} = \frac{(\frac{m}{n})^2 - 1}{(\frac{m}{n})^2 + 1} = \frac{m^2 - n^2}{m^2 + n^2}.$$

6. Now, if we remember that $x = a/c$ and $y = b/c$, where (a, b, c) is a Pythagorean triple, we get:

$$a = 2mn$$
$$b = m^2 - n^2$$
$$c = m^2 + n^2$$

— same formulas as in the previous approach.

Problems

1. Go through the Algebraic approach to deriving formulas for Pythagorean triples. Make sure you understand all the steps!

2. Find all Pythagorean triples such that $x < y < z < 30$.

3. Prove that for any Pythagorean triple (x, y, z) where $x^2 + y^2 = z^2$ the following holds:
   - (a) At least one of x, y is divisible by 3.
   - (b) At least one of x, y is divisible by 4.
   - (c) At least one of x, y, z is divisible by 5.

4. Prove that xy is divisible by 12.