The expression $1+\frac{1}{x}$ in the Solve It is equivalent to the *rational expression* $\frac{x+1}{x}$.

A **rational expression** is the quotient of two polynomials. You will find that, at different times, it is helpful to think of rational expressions as ratios, as fractions, or as quotients.

A rational expression is in **simplest form** if its numerator and denominator are polynomials that have no common divisors other than 1.

You simplify a rational expression by dividing out the common factors in the numerator and the denominator. Factoring the numerator and denominator will help you find the common divisors.

In simplest form

$\frac{x+1}{x-1}$ $\frac{x^2+3x+2}{x+3}$

Not in simplest form

$\frac{x}{x^2}$ $\frac{3(x-3)}{x-3}$ $\frac{x^2-x-6}{x^2+x-2}$

A rational expression and any simplified form must have the same domain in order to be equivalent.

$$\frac{x^2 - x - 6}{x^2 + x - 2} = \frac{(x-3)(x+2)}{(x-1)(x+2)} \text{ and } \frac{x-3}{x-1}, x \neq -2, \text{ are equivalent.}$$

In the example above, you must exclude –2 from the domain of $\frac{x-3}{x-1}$ because –2 is not in the domain of $\frac{x^2 - x - 6}{x^2 + x - 2}$. Note that this restriction is not evident from the simplified expression $\frac{x-3}{x-1}$.

be equivalent.

$x + 3$ is the simplest form of $\frac{(x+2)(x+3)}{x+2}$ but $x + 3$ and $\frac{(x+2)(x+3)}{x+2}$ are *not* equivalent. $\frac{(x+2)(x+3)}{x+2}$ and $x + 3$ need to have the same domain to be equivalent.

The domain of $x + 3$ is all real numbers. But the domain of $\frac{(x+2)(x+3)}{x+2}$ excludes –2.

Problem 1 Simplifying a Rational Expression

What is $\frac{x^2 + 7x + 10}{x^2 - 3x - 10}$ in simplest form? State any restrictions on the variable.

$$\frac{x^2 + 7x + 10}{x^2 - 3x - 10} = \frac{(x + 2)(x + 5)}{(x + 2)(x - 5)}$$ Factor the numerator and denominator.

$$= \frac{\cancel{(x + 2)}(x + 5)}{\cancel{(x + 2)}(x - 5)}$$ Divide out common factors.

$$= \frac{x + 5}{x - 5}$$ Simplify.

The simplified form is $\frac{x + 5}{x - 5}$ for $x \neq 5$ and $x \neq -2$. The restriction $x \neq -2$ is not evident from the simplified form, but is needed to prevent the denominator of the original expression from being zero.

You can use what you know about simplifying rational expressions when you multiply and divide them.

Problem 2 Multiplying Rational Expressions

What is the product $\frac{x^2+x-6}{x-5} \cdot \frac{x^2-25}{x^2+4x+3}$ in simplest form? State any restrictions on the variable.

$$\frac{x^2+x-6}{x-5} \cdot \frac{x^2-25}{x^2+4x+3}$$

$$= \frac{(x+3)(x-2)}{x-5} \cdot \frac{(x+5)(x-5)}{(x+3)(x+1)}$$ Factor all polynomials.

$$= \frac{\cancel{(x+3)}(x-2)}{\cancel{x-5}} \cdot \frac{(x+5)\cancel{(x-5)}}{\cancel{(x+3)}(x+1)}$$ Divide out common factors.

$$= \frac{(x-2)(x+5)}{x+1}$$ Simplify.

The product is $= \frac{(x-2)(x+5)}{x+1}$ and $x \neq -3$, $x \neq 5$, and $x \neq -1$. The restrictions $x \neq -3$ and $x \neq 5$ are not evident from the simplified form, but are needed to prevent the denominators in the original product from being zero.

To divide rational expressions, you multiply by the reciprocal of the divisor, just as you do when you divide rational numbers. To identify the restrictions on the variable for a quotient, you must find the restrictions from any denominator used in the calculation. This means you must find the restrictions on not just the denominators of the dividend and the divisor, but also on the *numerator* of the divisor.

Problem 3 Dividing Rational Expressions

What is the quotient $\frac{2-x}{x^2+2x+1} \div \frac{x^2+3x-10}{x^2-1}$ in simplest form? State any restrictions on the variable.

THINK	WRITE
To divide, you multiply by the reciprocal.	$\frac{2 - x}{x^2 + 2x + 1} \div \frac{x^2 + 3x - 10}{x^2 - 1}$ $= \frac{2 - x}{x^2 + 2x + 1} \cdot \frac{x^2 - 1}{x^2 + 3x - 10}$
The expressions may have common factors. So, factor the numerators and denominators.	$= \frac{2 - x}{(x + 1)(x + 1)} \cdot \frac{(x + 1)(x - 1)}{(x + 5)(x - 2)}$
Factor -1 from $(2 - x)$ to get a second $(x - 2)$.	$= \frac{-1\,(x - 2)}{(x + 1)(x + 1)} \cdot \frac{(x + 1)(x - 1)}{(x + 5)(x - 2)}$
Divide out common factors.	$= \frac{-1\,\cancel{(x - 2)}}{\cancel{(x + 1)}(x + 1)} \cdot \frac{\cancel{(x + 1)}(x - 1)}{(x + 5)\cancel{(x - 2)}}$
Rewrite the remaining factors.	$= \frac{-1\,(x - 1)}{(x + 1)(x + 5)}$
Identify the restrictions from the denominator of the simplified expression and from any other denominator used.	$x \neq -1, x \neq -5, x \neq 1, x \neq 2$

Problem 4 Using Rational Expressions to Solve a Problem

Construction Your community is building a park. It wants to fence in a play space for toddlers. It wants the maximum area for a given amount of fencing. Which shape, a *square* or a *circle*, provides a more efficient use of fencing?

One measure of efficiency is the ratio of *area fenced* to *fencing used*, or area to perimeter. Which of the two shapes has the greater ratio?

Square

$\text{Area} = s^2$ Define area and perimeter.

$\text{Perimeter} = 4s$

$s = \frac{P}{4}$ Express s and r in terms of a common variable, P.

$\frac{\text{Area}}{\text{Perimeter}} = \frac{s^2}{P}$ Write the ratios.

$= \frac{\left(\frac{P}{4}\right)^2}{P}$ Substitute for s and r.

$= \frac{P}{16}$ Simplify.

Circle

$\text{Area} = \pi r^2$ $\text{Perimeter} = 2\pi r$	Define area and perimeter.
$r = \dfrac{P}{2\pi}$	Express *s* and *r* in terms of a common variable, *P*.
$\dfrac{\text{Area}}{\text{Perimeter}} = \dfrac{\pi r^2}{P}$	Write the ratios.
$= \dfrac{\left(\dfrac{P}{2\pi}\right)^2}{P}$	Substitute for *s* and *r*.
$= \dfrac{P}{4\pi}$	Simplify.

Since $\frac{P}{4\pi} > \frac{P}{16}$, a circle provides a more efficient use of fencing.

Check Assume $P = 40$ ft. The area of the circle is $\pi\left(\frac{40}{2\pi}\right)^2 \approx 127\ \text{ft}^2$. The area of the square is $\left(\frac{40}{4}\right)^2 = 100\ \text{ft}^2$. The area of the circle is greater.

1. Simplify the rational expression $\frac{4z - 12}{8z + 24}$.

State any restrictions on the variable.

A. $\frac{z - 3}{2(z + 3)}$; $z \neq -3$

B. $\frac{z - 3}{2(z + 3)}$; $z \neq 3$

C. $\frac{z - 3}{2(z + 3)}$; $z \neq \pm 3$

D. $\frac{z - 3}{2(z + 3)}$; $z \neq -3, 3, \text{ or } 0$

2. Simplify the rational expression $\frac{3x-3}{x^2-x}$.

State any restrictions on the variable.

- A. $\frac{3}{x}$; $x \neq 0$
- B. $\frac{3}{x}$; $x \neq 0$ or 1
- C. $\frac{3}{x-1}$; $x \neq 1$
- D. $\frac{3}{x-1}$; $x \neq 0$ or 1

3. Multiply $\frac{x^2 + 3x - 10}{x^2 + 4x - 12}$ and $\frac{3x + 18}{x + 3}$. State any restrictions on the variable.

A. $\frac{3(x+5)}{x+3}$; $x \neq -3$

B. $\frac{(x+3)(x+5)}{3(x+6)^2}$; $x \neq -6$

C. $\frac{(x+3)(x+5)}{3(x+6)^2}$; $x \neq -3$ or -6

D. $\frac{3(x+5)}{x+3}$; $x \neq -3, -6$, or 2

4. Divide $\frac{x^2 - 7x + 10}{x^2 - 8x + 15}$ by $\frac{4 - x^2}{x^2 + 3x - 18}$. State any restrictions on the variable.

- **A.** $-\frac{x + 6}{x + 2}$; $x \neq -6, -2, 2, 3$, or 5
- **B.** $-\frac{x + 6}{x + 2}$; $x \neq -2, 2, 3$, or 5
- **C.** $\frac{x + 6}{x + 2}$; $x \neq -6, -2, 3$, or 5
- **D.** $\frac{x + 6}{x + 2}$; $x \neq -2$

A • Practice

Simplify each rational expression. State any restrictions on the variables. SEE PROBLEM 1.

8. $-\frac{5x^3y}{15xy^3}$

9. $\frac{2x}{4x^2 - 2x}$

10. $\frac{6c^2 + 9c}{3c}$

11. $\frac{49 - z^2}{z + 7}$

12. $\frac{x^2 + 8x + 16}{x^2 - 2x - 24}$

13. $\frac{12 - x - x^2}{x^2 - 8x + 15}$

Multiply. State any restrictions on the variables.
SEE PROBLEM 2.

14. $\frac{4x^2}{5y} \cdot \frac{7y}{12x^4}$

15. $\frac{2x^4}{10y^{-2}} \cdot \frac{5y^3}{4x^3}$

16. $\frac{8y - 4}{10y - 5} \cdot \frac{5y - 15}{3y - 9}$

17. $\frac{2x + 12}{3x - 9} \cdot \frac{6 - 2x}{3x + 8}$

18. $\frac{x^2 - 4}{x^2 - 1} \cdot \frac{x + 1}{x^2 + 2x}$

19. $\frac{x^2 - 5x + 6}{x^2 - 4} \cdot \frac{x^2 + 3x + 2}{x^2 - 2x - 3}$

Divide. State any restrictions on the variables.
SEE PROBLEM 3.

20. $\frac{7x}{4y^3} \div \frac{21x^3}{8y}$

21. $\frac{3x^3}{5y^2} \div \frac{6y^{-3}}{5x^{-5}}$

22. $\frac{6x + 6y}{y - x} \div \frac{18}{5x - 5y}$

23. $\frac{3y - 12}{2y + 4} \div \frac{6y - 24}{8 + 4y}$

24. $\frac{x^2}{x^2 + 2x + 1} \div \frac{3x}{x^2 - 1}$

25. $\frac{y^2 - 5y + 6}{y^3} \div \frac{y^2 + 3y - 10}{4y^2}$

26. Industrial Design A storage tank will have a circular base of radius r and a height of r. The tank can be either cylindrical or hemispherical (half a sphere). SEE PROBLEM 4.

a. Write and simplify an expression for the ratio of the volume of the hemispherical tank to its surface area (including the base). For a sphere, $V = \frac{4}{3}\pi r^3$ and $\text{S.A.} = 4\pi r^2$.

b. Write and simplify an expression for the ratio of the volume of the cylindrical tank to its surface area (including the bases).

c. Compare the ratios of volume to surface area for the two tanks.

d. Compare the volumes of the two tanks.

e. Describe how you used these ratios to compare the volumes of the two tanks. Which measurement of the tanks determines the volumes?

B • Apply

Simplify each rational expression. State any restrictions on the variables.

27. $\dfrac{x^2 - 5x - 24}{x^2 - 7x - 30}$

28. $\dfrac{2y^2 + 8y - 24}{2y^2 - 8y + 8}$

29. $\dfrac{xy^3 - 9xy}{12xy^2 + 12xy - 144x}$

30. Open-Ended Write three rational expressions that simplify to $\dfrac{x}{x+1}$.

31. Think About a Plan A cereal company wants to use the most efficient packaging for their new product. They are considering a cylindrical-shaped box and a cube-shaped box. Compare the ratios of the volume to the surface area of the containers to determine which packaging will be more efficient.

- How can you measure the cereal box's efficiency?
- What formulas will you need to use to solve this problem?

Multiply or divide. State any restrictions on the variables.

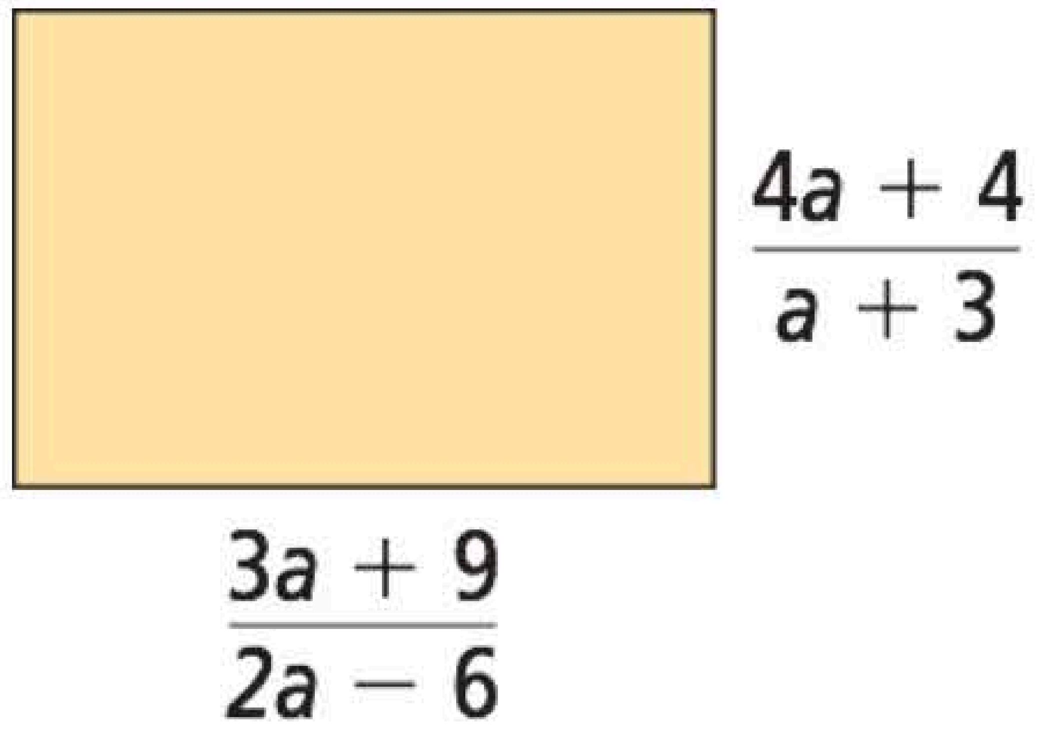
32. $\dfrac{6x^3 - 6x^2}{x^4 + 5x^3} \div \dfrac{3x^2 - 15x + 12}{2x^2 + 2x - 40}$

33. $\dfrac{2x^2 - 6x}{x^2 + 18x + 81} \cdot \dfrac{9x + 81}{x^2 - 9}$

34. $\dfrac{x^2 - x - 2}{2x^2 - 5x + 2} \div \dfrac{x^2 - x - 12}{2x^2 + 5x - 3}$

35. $\dfrac{2x^2 + 5x + 2}{4x^2 - 1} \cdot \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

36. a. Reasoning Write a simplified expression for the area of the rectangle.



b. Which parts of the expression do you analyze to determine the restrictions on a? Explain.

c. State all restrictions on a.

37. Manufacturing A toy company is considering a cube or sphere-shaped container for packaging a new product. The height of the cube would equal the diameter of the sphere. Compare the volume-to-surface area ratios of the containers. Which packaging will be more efficient? For a sphere, $SA = 4\pi r^2$.

Decide whether the given statement is *always, sometimes*, or *never* true.

38. Rational expressions contain exponents.

39. Rational expressions contain logarithms.

40. Rational expressions are undefined for values of the variables that make the denominator 0.

41. Restrictions on variables change when a rational expression is simplified.

Simplify. State any restrictions on the variables.

42. $\dfrac{\left(x^2-x\right)^2}{x(x-1)^{-2}\left(x^2+3x-4\right)}$

43. $\dfrac{2x+6}{(x-1)^{-1}(x^2+2x-3)}$

44. $\dfrac{54x^3y^{-1}}{3x^{-2}y}$

C • Challenge

45. a. Reasoning Simplify $\dfrac{\left(2x^n\right)^2-1}{2x^n-1}$, where x is an integer and n is a positive integer. (*Hint:* Factor the numerator.)

b. Use the result from part (a). Which part(s) of the expression can you use to show that the value of the expression is always odd? Explain.

Use the fact that $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{a}{b}\div\dfrac{c}{d}$ to simplify each rational expression. State any restrictions on the variables.

46. $\dfrac{\frac{8x^2y}{x+1}}{\frac{6xy^2}{x+1}}$

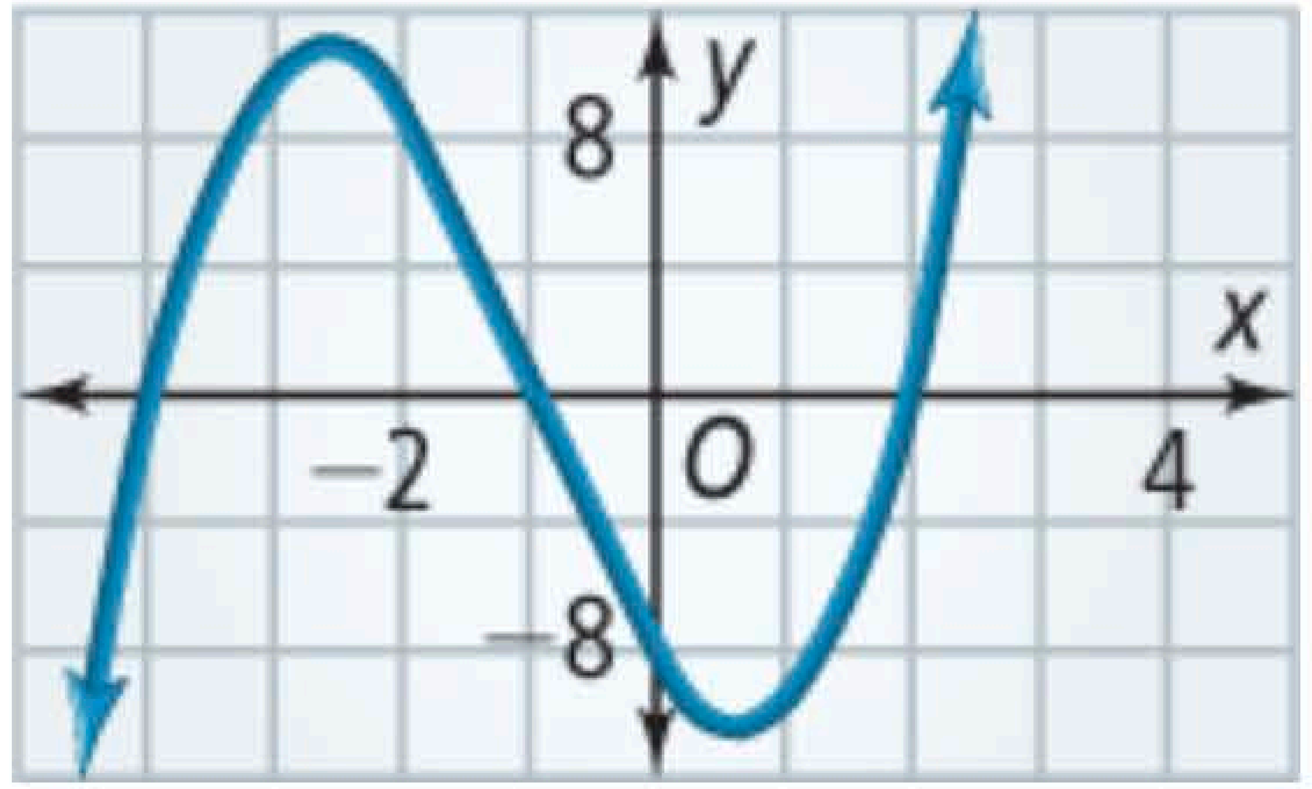
47. $\dfrac{\frac{3a^3b^3}{a-b}}{\frac{4ab}{b-a}}$

48. $\dfrac{\frac{9m+6n}{m^2n^2}}{\frac{12m+8n}{5m^2}}$

49. $\dfrac{\frac{x^2-1}{x^2-9}}{\frac{x^2+3x-4}{x^2+8x+15}}$

SAT/ACT

50. Which function is graphed below?



A. $y = (x+4)(x-1)(x+2)$

B. $y = (x-4)(x-1)(x+2)$

C. $y = (x-4)(x+1)(x-2)$

D. $y = (x+4)(x+1)(x-2)$

51. Which function generates the table of values?

x	y
$\frac{1}{2}$	-1
1	0
2	1
4	2

A. $y = \log_{\frac{1}{2}} x$

B. $y = -\log_2 x$

C. $y = \log_2 x$

D. $y = \left(\frac{1}{2}\right)^x$

52. Which expression equals $\frac{x}{x^2 - 2x - 3} \cdot \frac{2x - 6}{x^2 - 4x + 3}$?

A. $\frac{2x - 1}{(x - 1)(x + 3)(x + 1)}$

B. $\frac{2x + 1}{(x - 1)(x + 1)(x - 3)}$

C. $\frac{2x}{(x - 1)(x + 1)(x - 3)}$

D. $\frac{2x}{(x + 3)(x - 1)(x + 1)}$

Short Response

53. What is the solution of the equation $3^{-x} = \frac{1}{243}$?