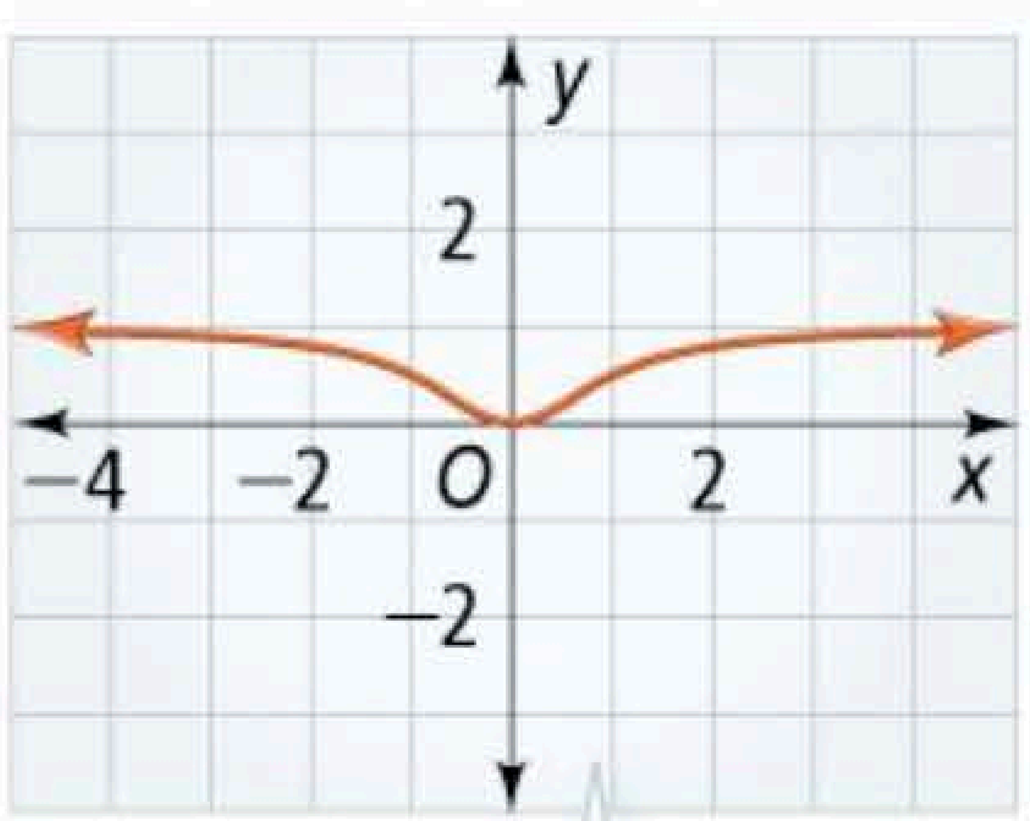
You use a ratio of polynomial functions to form a *rational function*, like $y = \frac{x+3}{x+16}$.

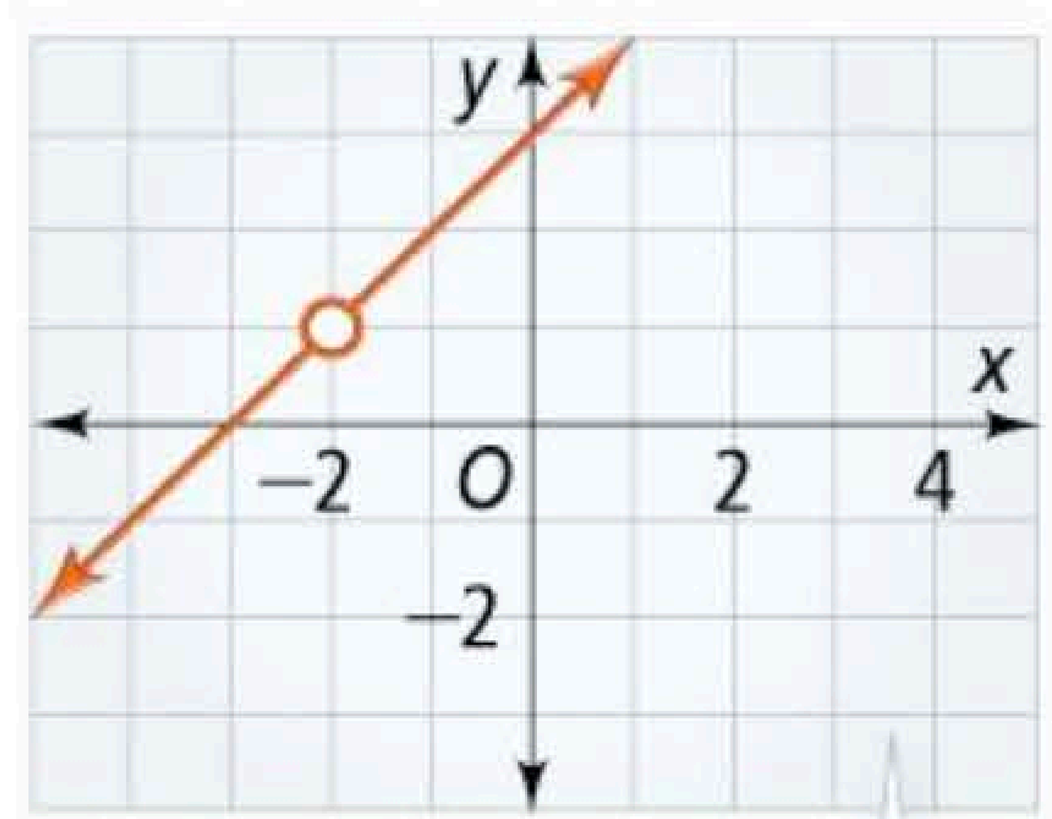
A **rational function** is a function that you can write in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions. The domain of $f(x)$ is all real numbers except those values for which $Q(x) = 0$.

The graph of a rational function can be *continuous* or *discontinuous*. The graphs of three rational functions are shown below.

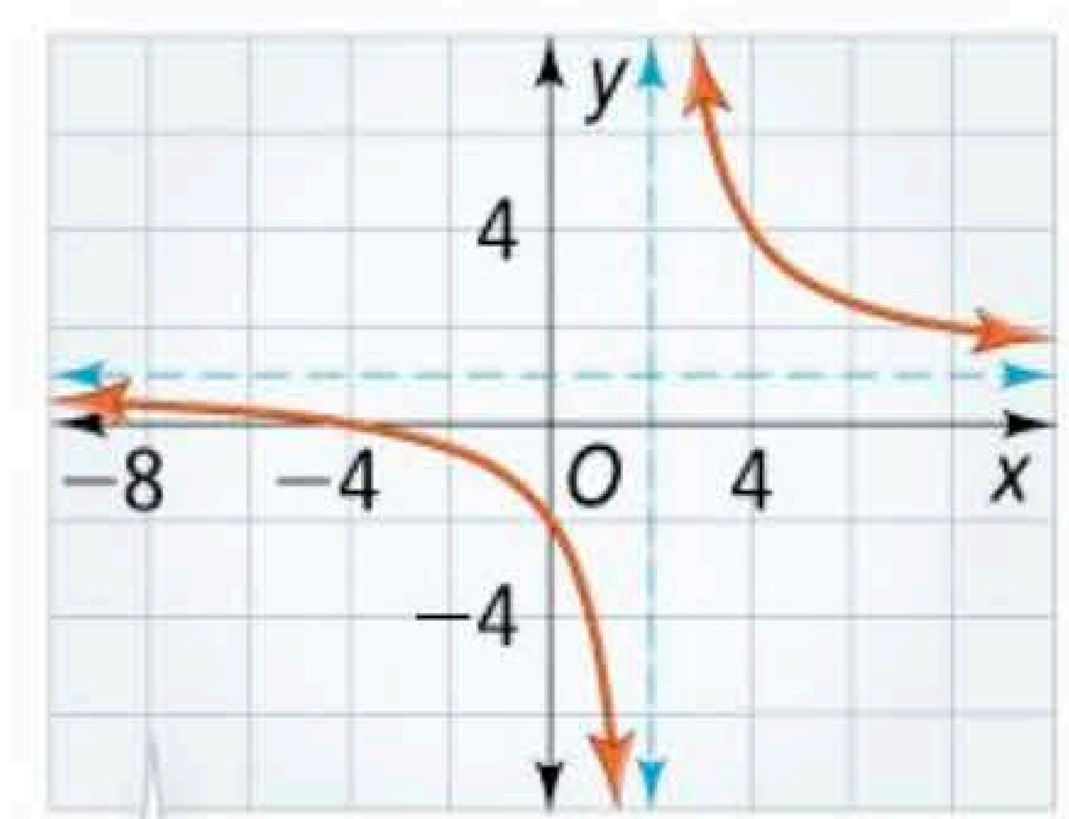
$y = \frac{x^2}{x^2+1}$




$y = \frac{(x+3)(x+2)}{(x+2)}$




$y = \frac{x+4}{x-2}$




**Continuous graph**

**Discontinuous graph**

## TAKE NOTE Key Concept

### Point of Discontinuity

If $a$ is a real number for which the denominator of a rational function $f(x)$ is zero, then $a$ is not in the domain of $f(x)$. The graph of $f(x)$ is not continuous at $x = a$ and the function has a **point of discontinuity** at $x = a$. A point of discontinuity can be removable or non-removable.

- A **removable discontinuity** is a point of discontinuity $a$ of a function $f$ that you can remove by redefining $f$ at $x = a$. Doing so fills in a hole in the graph of $f$ with the point $(a, f(a))$.

- A **non-removable discontinuity** is a point of discontinuity $a$ of a function $f$ that is not removable. It represents a break in the graph of $f$ where you cannot redefine $f$ to make the graph continuous.

When you are looking for discontinuities, it is helpful to factor the numerator and denominator as a first step. The factors of the denominator will reveal the points of discontinuity. The discontinuity caused by $(x - a)^n$ in the denominator is removable if the numerator also has $(x - a)^n$ as a factor.

## Problem 1 Finding Points of Discontinuity

**What are the domain and points of discontinuity of each rational function? Are the points of discontinuity removable or non-removable? What are the *x*- and *y*-intercepts?**

**A** $y = \frac{x+3}{x^2-4x+3}$

Factor the numerator and denominator to check for common factors.

$$y = \frac{x+3}{x^2-4x+3} = \frac{x+3}{(x-3)(x-1)}$$

The function is undefined where $x - 3 = 0$ and where $x - 1 = 0$, at $x = 3$ and $x = 1$. The domain of the function is the set of all real numbers except $x = 1$ and $x = 3$.

There are non-removable points of discontinuity at $x = 1$ and $x = 3$. The *x*-intercept occurs where the numerator equals 0, at $x = -3$.

To find the *y*-intercept, let $x = 0$ and simplify.

$$y = \frac{0+3}{(0-3)(0-1)} = \frac{3}{(-3)(-1)} = \frac{3}{3} = 1$$

**B** $y = \frac{x-5}{x^2+1}$

You cannot factor the numerator or the denominator. Also, there are no values of *x* that make the denominator 0. The domain of the function is all real numbers, and there are no discontinuities.

The *x*-intercept occurs where the numerator equals 0, at $x = 5$.

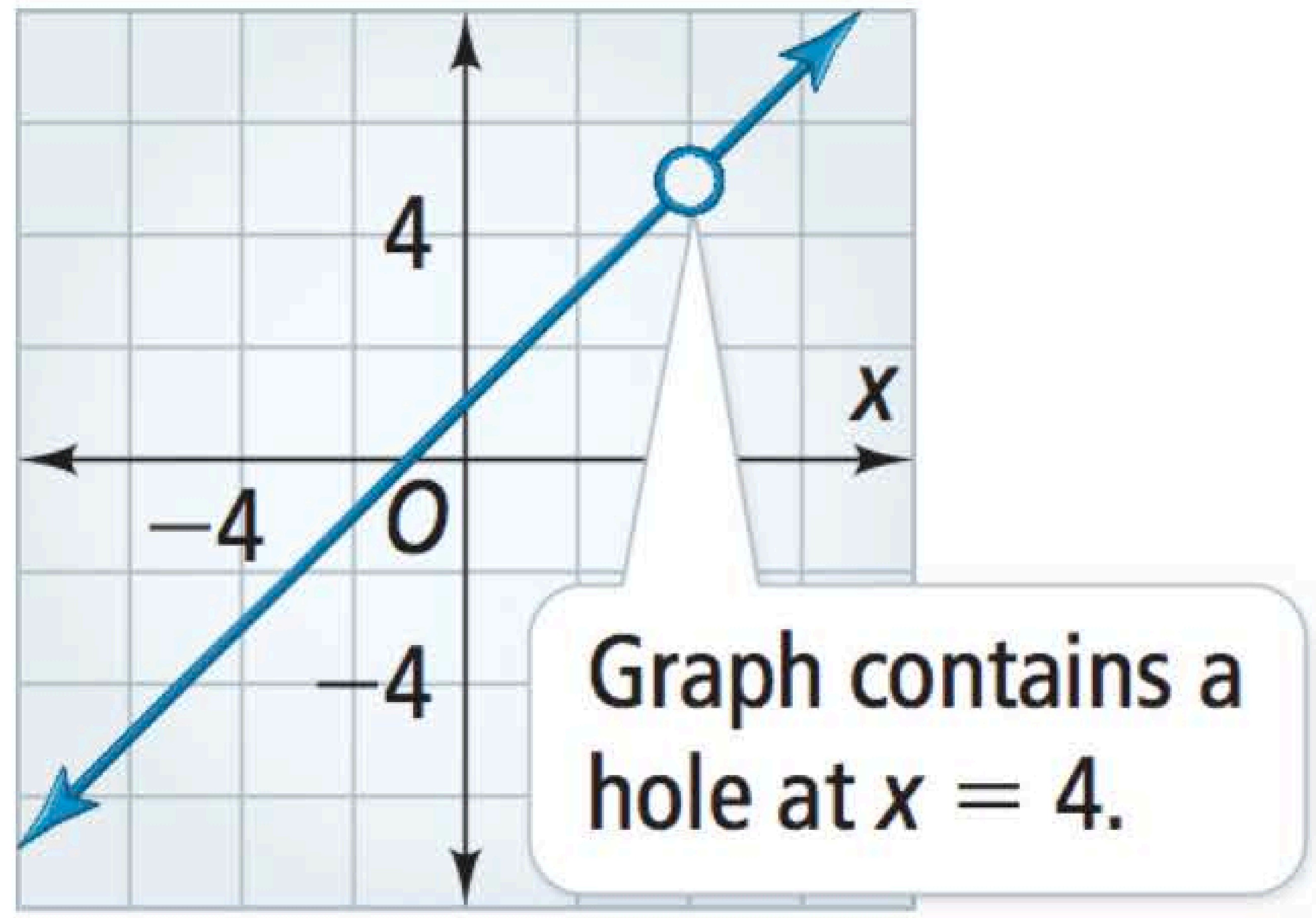
To find the *y*-intercept, let $x = 0$ and simplify: $y = \frac{0-5}{0^2+1} = \frac{-5}{1} = -5$

**C** $y = \dfrac{x^2 - 3x - 4}{x - 4}$

Factor the numerator and denominator: $y = \dfrac{x^2 - 3x - 4}{x - 4} = \dfrac{(x-4)(x+1)}{(x-4)}$

The function is undefined where $x - 4 = 0$, at $x = 4$. The domain of the function is the set of all real numbers except $x = 4$.

Because $y = x + 1$, except at $x = 4$, there is a removable discontinuity at $x = 4$.




At $x = 4$, $y = x + 1 = 4 + 1 = 5$, so you can redefine the function to remove the discontinuity.

$$y = \begin{cases} \dfrac{x^2 - 3x - 4}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4 \end{cases}$$

The $x$-intercept occurs where the numerator equals 0, at $x = -1$.

To find the $y$-intercept, let $x = 0$ and simplify.

$$y = \frac{0^2 - 3 \cdot 0 - 4}{0 - 4} = \frac{0 - 0 - 4}{-4} = \frac{-4}{-4} = 1$$

In Chapter 7, you learned that an asymptote is a line that a graph approaches as $x$ or $y$ increases in absolute value. If a rational function has a non-removable discontinuity at $x = a$, the graph of the rational function will have a vertical asymptote at $x = a$.

TAKE NOTE Key Concept

## Vertical Asymptotes of Rational Functions

The graph of the rational function $f(x) = \frac{P(x)}{Q(x)}$ has a vertical asymptote at each real zero of $Q(x)$ if $P(x)$ and $Q(x)$ have no common zeros. If $P(x)$ and $Q(x)$ have $(x - a)^m$ and $(x - a)^n$ as factors, respectively and $m < n$, then $f(x)$ also has a vertical asymptote at $x = a$.

In Chapter 7, you learned that an asymptote is a line that a graph approaches as $x$ or $y$ increases in absolute value. If a rational function has a non-removable discontinuity at $x = a$, the graph of the rational function will have a vertical asymptote at $x = a$.

TAKE NOTE Key Concept

## Vertical Asymptotes of Rational Functions

Consider the function $f(x) = \frac{(x+2)^2}{(x+2)^3}$. The function can be simplified to be $f(x) = \frac{1}{(x+2)}$, so the graph of the function will have a vertical asymptote at $x = -2$.

## Problem 2 Finding Vertical Asymptotes

**What are the vertical asymptotes for the graph of $y = \frac{x+1}{(x-2)(x-3)}$?**

Since 2 and 3 are zeros of the denominator and neither is a zero of the numerator, the lines $x = 2$ and $x = 3$ are vertical asymptotes.

---

### TAKE NOTE Key Concept

#### Horizontal Asymptote of a Rational Function

To find the horizontal asymptote of the graph of a rational function, compare the degree of the numerator $m$ to the degree of the denominator $n$.

If $m < n$, the graph has horizontal asymptote $y = 0$ (the $x$-axis).

If $m > n$, the graph has no horizontal asymptote.

If $m = n$, the graph has horizontal asymptote $y = \frac{a}{b}$ where $a$ is the coefficient of the term of greatest degree in the numerator and $b$ is the coefficient of the term of greatest degree in the denominator.

## Problem 3 Finding Horizontal Asymptotes

**What is the horizontal asymptote for the rational function?**

**A** $y = \dfrac{2x}{x - 3}$

The degree of the numerator and denominator are the same.

The horizontal asymptote is $y = \frac{2}{1}$ or $y = 2$.

**B** $y = \dfrac{x - 2}{x^2 - 2x - 3}$

The degree of the numerator is less than the degree of the denominator.

The horizontal asymptote is $y = 0$.

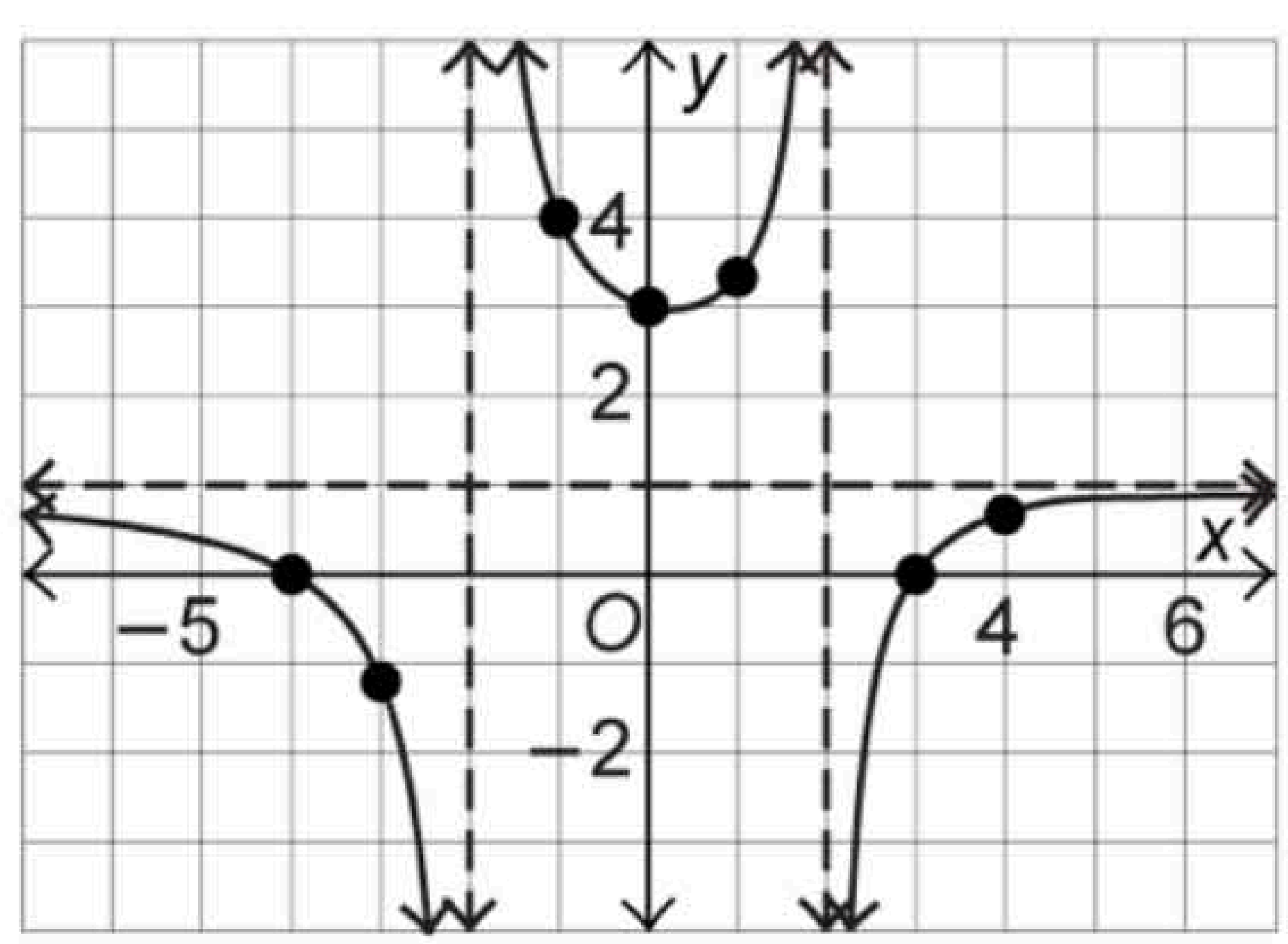
**C** $y = \dfrac{x^2}{2x - 5}$

The degree of the numerator is greater than the degree of the denominator.

There is no horizontal asymptote.

## Problem 4 Graphing a Rational Function

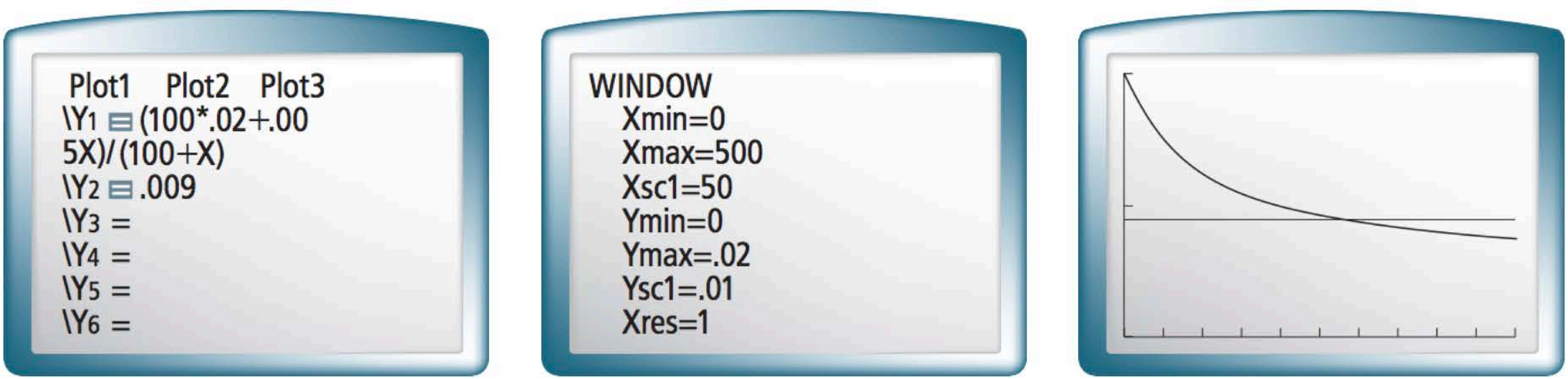
**What is the graph of the rational function $y = \dfrac{x^2 + x - 12}{x^2 - 4}$?**

| THINK | WRITE |
|---|---|
| The degrees of the numerator and denominator are equal. | $y = \dfrac{x^2 + x - 12}{x^2 - 4}$<br>horizontal asymptote: $y = \frac{1}{1} = 1$ |
| Factor the numerator and the denominator. They have no common factor. The graph has no holes. It has two vertical asymptotes at the zeros of the denominator. | $y = \dfrac{(x + 4)(x - 3)}{(x + 2)(x - 2)}$<br>vertical asymptotes: $x = -2$, $x = 2$ |
| Find the *x*- and *y*-intercepts. The *x*-intercepts occur where $y = 0$. The *y*-intercepts occur where $x = 0$. | When the numerator equals zero, $y = 0$. *x*-intercepts: $(-4, 0)$ and $(3, 0)$<br>$y = \dfrac{(0 + 4)(0 - 3)}{(0 + 2)(0 - 2)} = 3$<br>*y*-intercept: $(0, 3)$ |
| Find a few more points on the graph. | More points on the graph:<br>$\left(-3, -\frac{6}{5}\right)$, $(-1, 4)$, $\left(1, \frac{10}{3}\right)$ and $\left(4, \frac{2}{3}\right)$ |
| Graph the asymptotes. Then plot the intercepts and additional points. Use the points to sketch the graph. |  |

## Problem 5 Using a Rational Function GRIDDED RESPONSE

**Chemistry** **You work in a pharmacy that mixes different concentrations of saline solutions for its customers. The pharmacy has a supply of two concentrations, 0.5% and 2%. The function $y = \frac{(100)(0.02) + x(0.005)}{100 + x}$ gives the concentration of the saline solution after adding $x$ milliliters of the 0.5% solution to 100 milliliters of the 2% solution. How many milliliters of the 0.5% solution must you add for the combined solution to have a concentration of 0.9%?**

**Step 1** Use a graphing calculator to graph $\mathbf{Y1} = \frac{(100)(0.02) + x(0.005)}{100 + x}$ and $\mathbf{Y2} = 0.009$.



**Step 2** Find the point of intersection of the two functions.

Graphic Solution

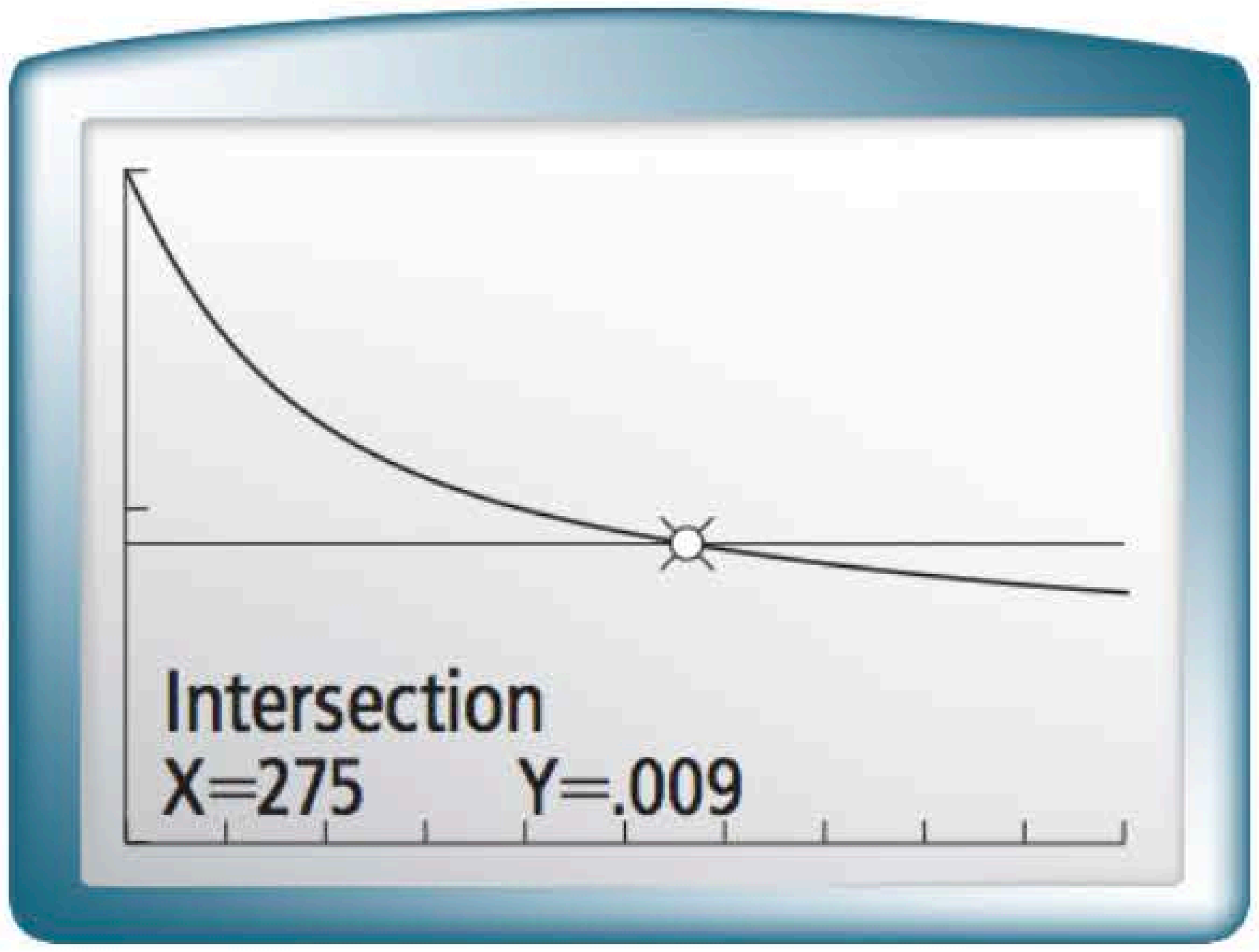
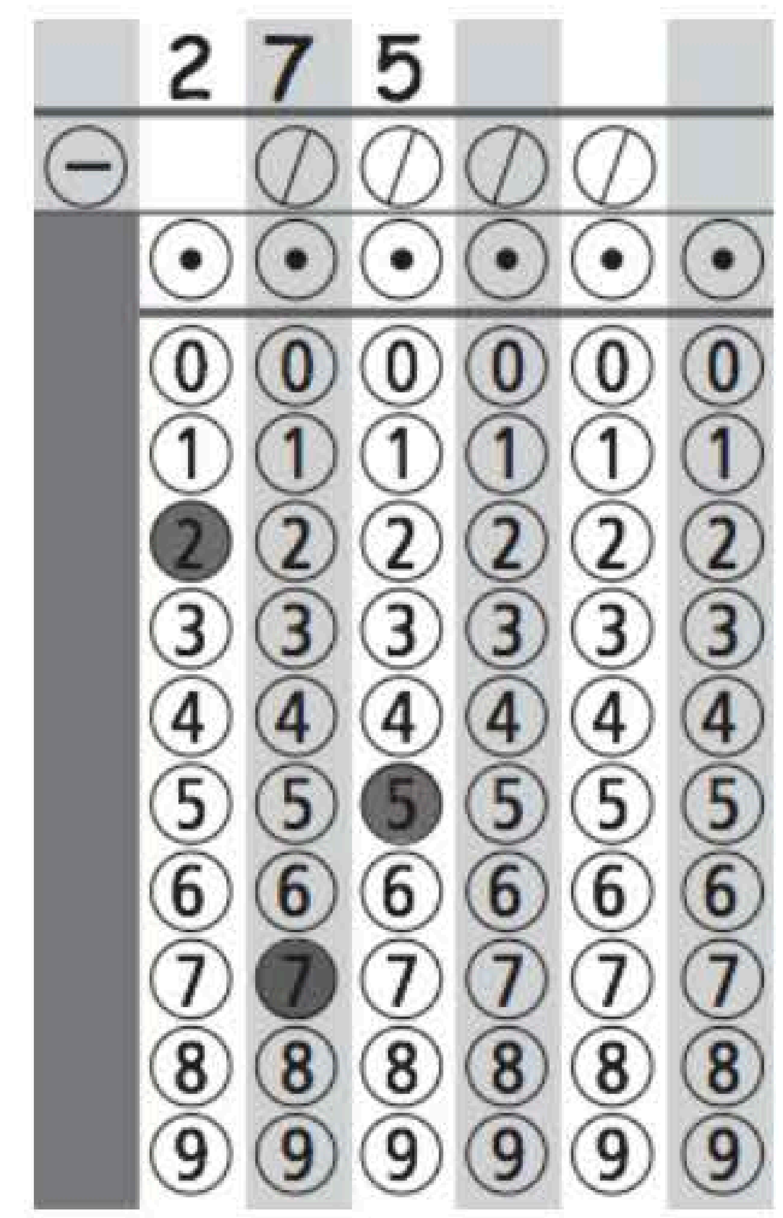



Table Solution

| X | Y1 | Y2 |
|---|---|---|
| 150 | .011 | .009 |
| 175 | .01045 | .009 |
| 200 | .01 | .009 |
| 225 | .00962 | .009 |
| 250 | .00929 | .009 |
| 275 | .009 | .009 |
| 300 | .00875 | .009 |

X=275

You should add 275 mL of the 0.5% solution to get a 0.9% solution. Write 275 in the grid.




**Check** $y = \dfrac{(100)(0.02) + x(0.005)}{100 + x}$

$y \stackrel{?}{=} \dfrac{(100)(0.02) + (275)(0.005)}{100 + 275}$ Substitute 275 for $x$.

$y \stackrel{?}{=} \dfrac{2 + 1.375}{375}$

$y = 0.009$ ✓

## A • Practice

**Find the domain, points of discontinuity, and *x*- and *y*-intercepts of each rational function. Determine whether the discontinuities are removable or non-removable.** SEE PROBLEM 1.

13. $y = \frac{2x^2 + 5}{x^2 - 2x}$

14. $y = \frac{x^2 + 2x}{x^2 + 2}$

15. $y = \frac{3x - 3}{x^2 - 1}$

16. $y = \frac{6 - 3x}{x^2 - 5x + 6}$

**Find the vertical asymptotes and holes for the graph of each rational function.** SEE PROBLEM 2.

17. $y = \frac{3}{x + 2}$

18. $y = \frac{x + 5}{x + 5}$

19. $y = \frac{x + 3}{(2x + 3)(x - 1)}$

20. $y = \frac{(x + 3)(x - 2)}{(x - 2)(x + 1)}$

21. $y = \frac{x^2 - 4}{x + 2}$

22. $y = \frac{x + 5}{x^2 + 9}$

**Find the horizontal asymptote of the graph of each rational function.** SEE PROBLEM 3.

23. $y = \frac{5}{x + 6}$

24. $y = \frac{x + 2}{2x^2 - 4}$

25. $y = \frac{x + 1}{x + 5}$

26. $y = \frac{x^2 + 2}{2x^2 - 1}$

27. $y = \frac{5x^3 + 2x}{2x^5 - 4x^3}$

28. $y = \frac{3x - 4}{4x + 1}$

**Sketch the graph of each rational function.**

SEE PROBLEM 4.

29. $y = \dfrac{x^2 - 4}{3x - 6}$

30. $y = \dfrac{4x}{x^3 - 4x}$

31. $y = \dfrac{x + 4}{x - 4}$

32. $y = \dfrac{x(x + 1)}{x + 1}$

33. $y = \dfrac{x + 6}{(x - 2)(x + 3)}$

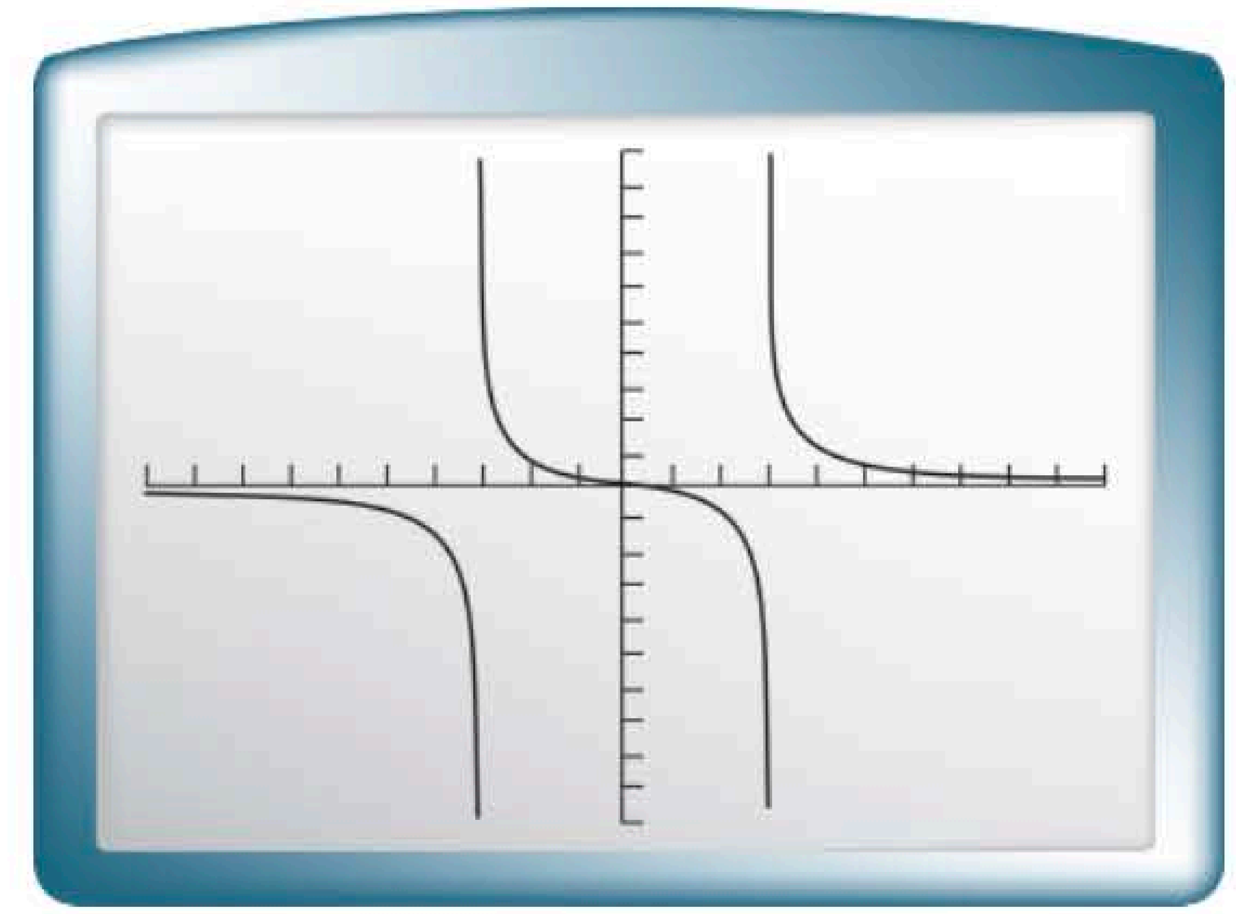
34. $y = \dfrac{3x}{(x + 2)^2}$

35. **Pharmacology** How many milliliters of the 0.5% solution must be added to the 2% solution to get a 0.65% solution? Use the rational function given in Problem 5. SEE PROBLEM 5.

## B • Apply

**Find the vertical and horizontal asymptotes, if any, of the graph of each rational function.**

36.



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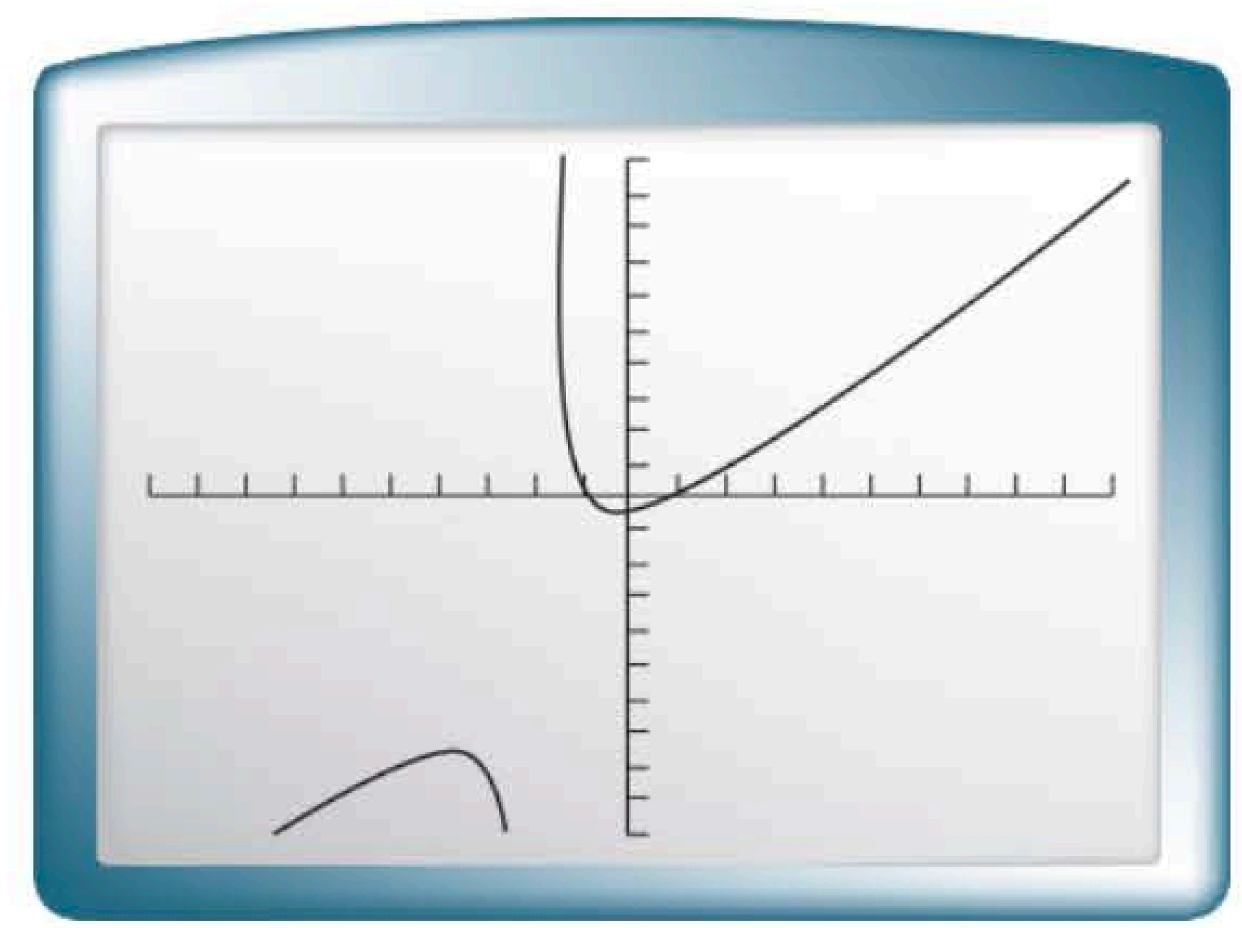
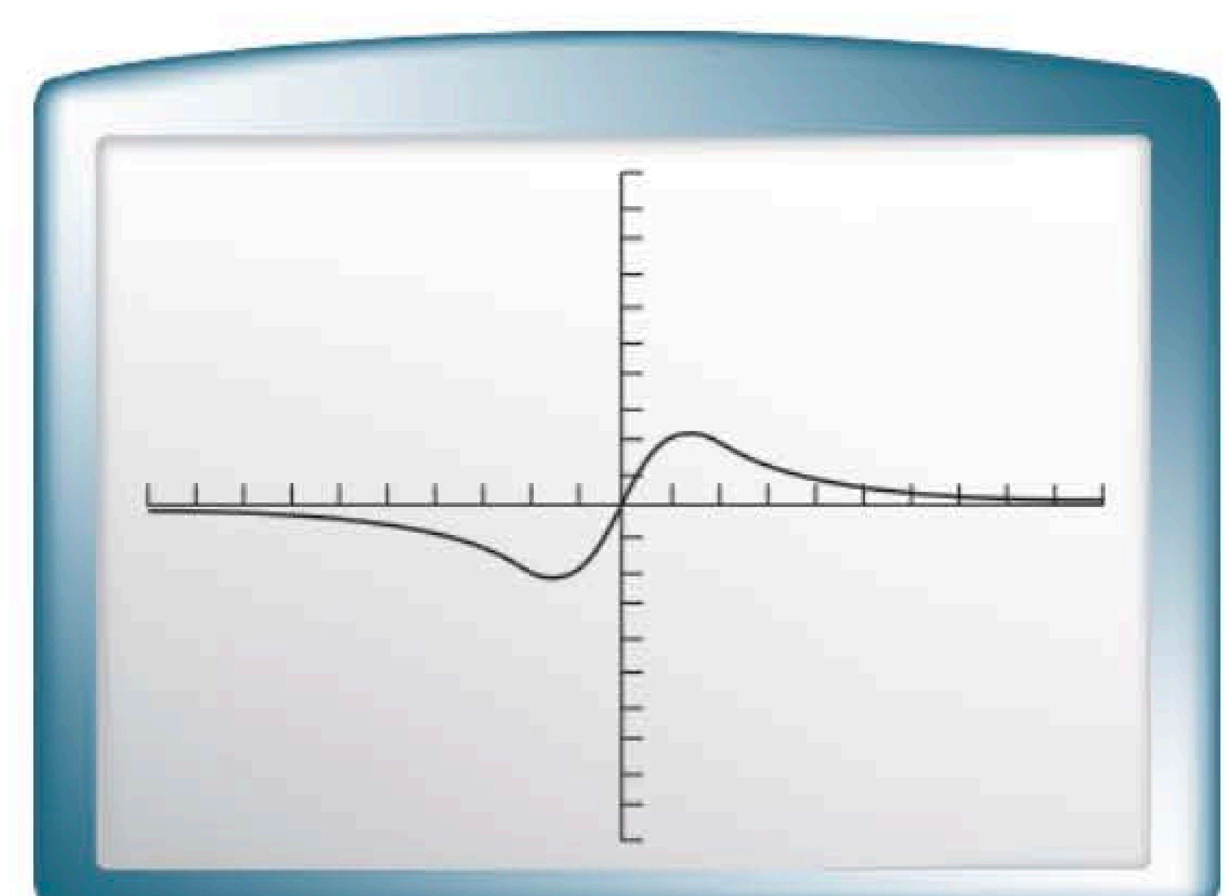


38.



**Sketch the graph of each rational function.**

**42.** $y = \frac{2x+3}{x-5}$

**43.** $y = \frac{x^2+6x+9}{x+3}$

**44.** $y = \frac{4x^2-100}{2x^2+x-15}$

**45.** $y = -\frac{x}{(x-1)^2}$

**46. Business** CDs can be manufactured for $.19 each. The development cost is $210,000. The first 500 discs are samples and will not be sold.

**a.** Write a function for the average cost of a disc that is not a sample. Graph the function.

**b.** What is the average cost if 5000 discs are produced? If 15,000 discs are produced?

**c.** How many discs must be produced to bring the average cost under $10?

**d.** What are the vertical and horizontal asymptotes of the graph of the function?

**47. Writing** Describe the conditions that will produce a rational function with a graph that has no vertical asymptotes.

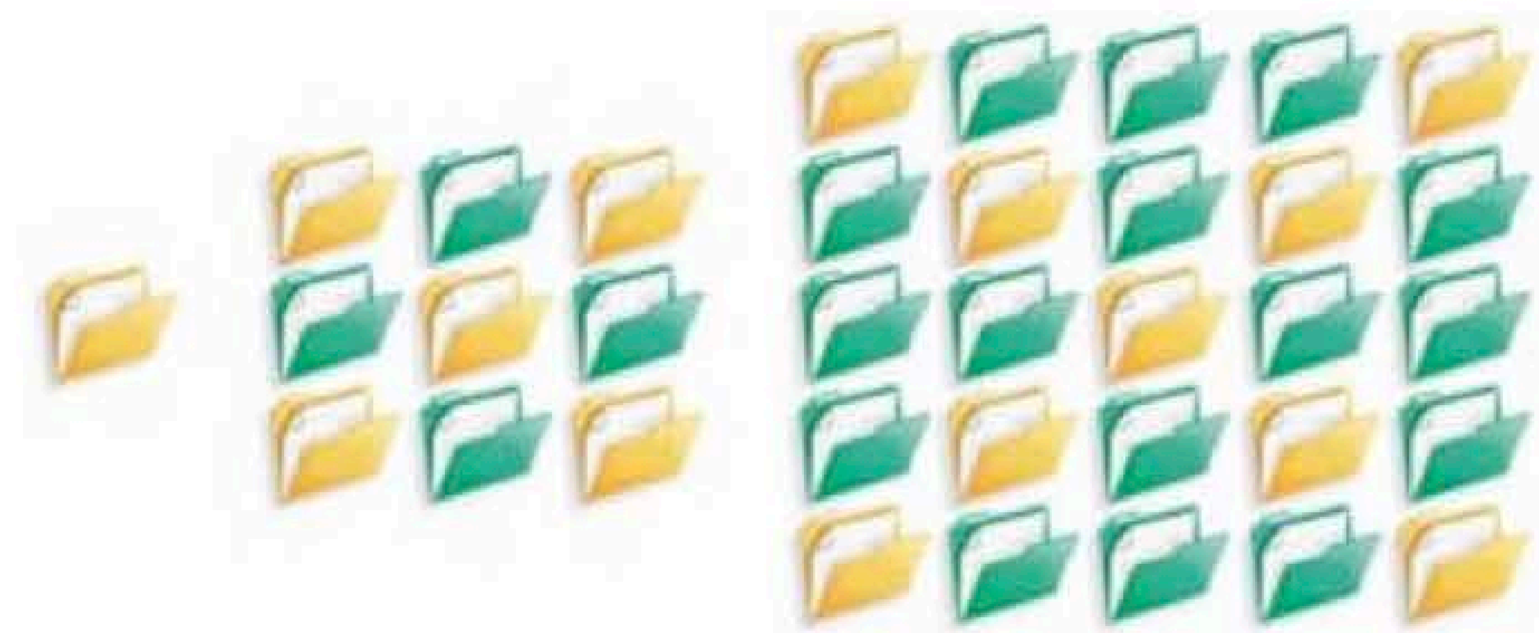
## C • Challenge

**48. Reasoning** Look for a pattern in the sequence of file folders.

**a.** Write a model for the number of yellow folders $Y(n)$ at each step $n$.

**b.** Write a model for the number of green folders $G(n)$ at each step $n$.

**c.** Write a model for the ratio of $Y(n)$ to $G(n)$. Use it to predict the ratio of yellow folders to green folders in the next figure. Verify your answer.



**49.** Write a rational function with the following characteristics.

**a.** Vertical asymptotes at $x = 1$ and $x = -3$, horizontal asymptote at $y = 1$, zeros at 3 and 4

**b.** Vertical asymptotes at $x = 0$ and $x = 3$, horizontal asymptote at $y = 0$, a zero at $-4$

**c.** Vertical asymptotes at $x = -2$ and $x = 2$, horizontal asymptote at $y = 3$, only one zero at $-1$

## SAT/ACT

**50.** What is the $x$-coordinate of the hole in the graph of $y = \dfrac{x^2 - 9}{2x^2 - x - 15}$?

**51.** Suppose $z$ varies directly with $x$ and inversely with $y$. If $z$ is 1.5 when $x$ is 9 and $y$ is 4, what is $z$ when $x$ is 6 and $y$ is 0.5?

**52.** What is the $y$-coordinate of the vertex of the parabola $y = -3(x - 4)^2 + 5$?

**53.** What is the real solution of $54x^3 - 16 = 0$ written as a fraction?

**54.** Using the Change of Base Formula, what is the value of $\log_7 15$ rounded to the nearest hundredth?