MATH 7
ASSIGNMENT 22: EUCLIDEAN GEOMETRY III

APR 5, 2020

1. Congruence

In general, two figures are called *congruent* if they have same shape and size.

- For line segments, it means that they have the same length: $\overline{AB} \cong \overline{CD}$ is the same as $AB = CD$.
- For angles, it means that they have the same measure: $\angle A \cong \angle B$ is the same as $m\angle A = m\angle B$.
- For triangles, it means that the corresponding sides are equal and corresponding angles are equal: $\triangle ABC \cong \triangle A'B'C'$ is the same as
  $AB = A'B'$, $BC = B'C'$, $AC = A'C'$, $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$.

Note that for triangles, the notation $\triangle ABC \cong \triangle A'B'C'$ not only tells that these two triangles are congruent, but also shows which vertex of the first triangle corresponds to which vertex of the second one. For example, $\triangle ABC \cong \triangle PQR$ is not the same as $\triangle ABC \cong \triangle QPR$.

2. Congruence tests for triangles

By definition, to check that two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalitites. However, it turns out that in fact, we can do with fewer checks.

- Angle-Side-Angle Congruence Axiom (ASA): If $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $AB = A'B'$, then $\triangle ABC \cong \triangle A'B'C'$.
- Side-Side-Side Congruence Axiom (SSS): If $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.
- Side-Angle-Side Congruence Axiom (SAS): If $AB = A'B'$, $AC = A'C'$ and $m\angle A = m\angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.

3. Isosceles triangles

A triangle is *isosceles* if two of its sides have equal length. The two sides of equal length are called *legs*; the point where the two legs meet is called the *apex* of the triangle; the other two angles are called the *base angles* of the triangle; and the third side is called the *base*.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 11. *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$. Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.*

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the *altitude* from A is perpendicular to BC (it exists and is unique by Theorem 7); the *median* from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the *angle bisector* bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem 12. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*

Homework

1. Let $\triangle ABC$ be such that all sides have equal length. Prove that then $m\angle A = m\angle B = m\angle C = 60^\circ$. [Such a triangle is called *equilateral.*]

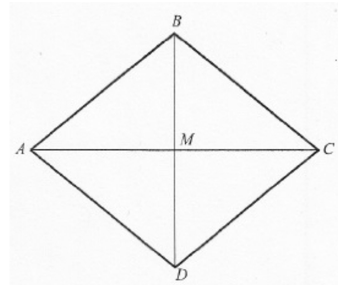
2. Prove theorem 11.

3. Let $ABCD$ be a quadrilateral such that $AB = BC = CD = AD$ (such a quadilateral is called a *rhombus*). Let M be the intersection point of AC and BD.

(a) Show that $\triangle ABC \cong \triangle ADC$

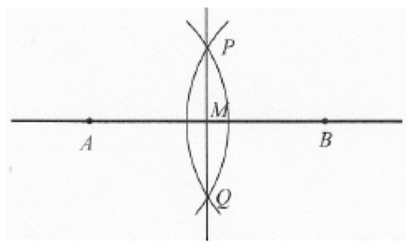
(b) Show that $\triangle AMB \cong \triangle AMD$

(c) Show that the diagonals are perpendicular and that the point M is the midpoint of each of the diagonals.



4. The following method explains how one can find the midpoint of a segment AB using a ruler and compass:

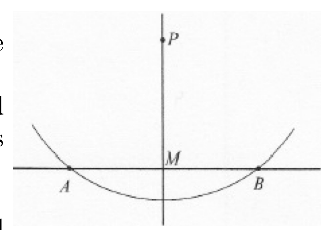
- Choose radius r (it should be large enough) and draw circles of radius r with centers at A and B.
- Denote the intersection points of these circles by P and Q. Draw a line $\overleftrightarrow{PQ}$.
- Let M be the intersection point of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. Then M is the midpoint of AB.



Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of AB? You can use the defining property of the circle: for a circle of radius r, the distance from any point on this circle to the center is exactly r.[Hint: use the previous problem]

5. The following method explains how one can construct a perpendicular from a point P to line l using a ruler and compass:

- Choose radius r (it should be large enough) and draw circle of radius r with center at P.
- Let A, B be the intersection points of this circle with l. Find the midpoint M of AB (using the method of the previous problem). Then $\overleftrightarrow{MP} \perp l$.



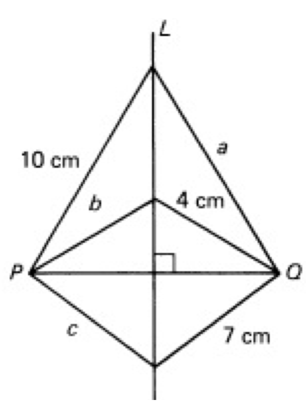
Can you justify this method, i.e., prove that so constructed $\overleftrightarrow{MP}$ will indeed be perpendicular to l?

6. What is the angle between the hour hand and minute hand of a clock at 11:10 AM?

Extra Problems

7. In the triangle $\triangle ABC$, let M be a point on side AB. Prove that $m\angle BMC > m\angle A$.

8. Line L is the perpendicular bisector of PQ. Find a, b and c.



9. Line K is the perpendicular bisector of PQ, line L is the perpendicular bisector of QM. Show that $OP = OM$.