MATH 6: EUCLIDEAN GEOMETRY 8

Coordinate Geometry

An *xy-coordinate plane*, often called the xy plane or the coordinate plane, is a plane with a pair of perpendicular lines, called the x-axis and the y-axis, together with a unit distance, called 1, which serve as references that we can use to measure the position of any point in the plane.

The xy plane is extremely useful for maps, blueprints, graphics design, and other applications - but even in the realm of theory, it has some curious theorems of its own.

Key Definitions

Here are the key definitions of coordinate geometry:

- The intersection of the x and y axes is called the *origin*.
- The distance of a point from the y-axis is called its *x-coordinate*, and the distance of the point from the x-axis is called its *y-coordinate*. It is necessary to distinguish points on different sides of the axis, so we will choose one side of each axis to be the *negative* side, and coordinates of points in these regions will be negative.
- The x and y coordinates of a point together are simply called its *coordinates*, and are written as an ordered pair with the x-coordinate first, like this: $(1, 2)$.
- An *equation* is an algebraic statement relating the x and y coordinates of a point. An example of an equation is $x = y$, which states that the x-coordinate of a point should be equal to the y-coordinate. It is a logical statement: given any point, the equation can be true or it can be false. For example, $x = y$ is true for $(1, 1)$ but not for $(1, 2)$.
- A *graph* of an equation is the figure of all the points in the xy-plane for which the given equation holds true.
- Given a line l and a right triangle whose hypotenuse is a segment of l and whose legs are parallel to the x and y axes, the ratio of the length of the y-parallel leg to the length of the x-parallel leg is called the *slope* of the line.

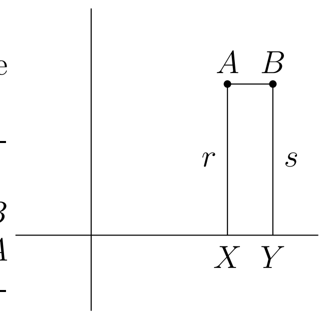
Note that it is conventional to draw the x axis as a horizontal line and the y axis as a vertical line, with the negative sides below the x axis and to the left of the y axis.

Key Theorems

Here are some important theorems of coordinate geometry. You may refer to them as the *Coordinate Theorems*.

Theorem 1. *Let A and B be two points in the xy plane with coordinates (a, r) and (b, s). If $r = s$ (the y-coordinates are equal), then AB is parallel to the x-axis, and has length $|b - a|$. Similarly, if $a = b$ (the x-coordinates are equal), then AB is parallel to the y-axis, and has length $|s - r|$.*

Proof. Suppose that A and B have the same y-coordinate, so that $r = s$. Let X be the foot of the perpendicular from A to the x-axis, and Y be the foot of the perpendicular from B to the x-axis, as seen in the figure. Then $AX = r$ and $BY = s$, thus $AX = BY$ because $r = s$.
We know that because AX and BY are both perpendicular to the x-axis, that they must be parallel to each other.
Thus $AXYB$ is a quadrilateral with one pair of opposite sides congruent and parallel, thus it is a parallelogram.
Therefore, AB is parallel to the x-axis. Also, the length of AB is equal to the length of the segment XY; noticing that X and A must be the the same distance from the y-axis, we find that the coordinates of X are $(a, 0)$ - then, similarly, the coordinates of Y are $(b, 0)$, and it's not hard to see from the definitions that the segment XY then has length $b - a$.
The proof for the y-axis parallel case is similar. □



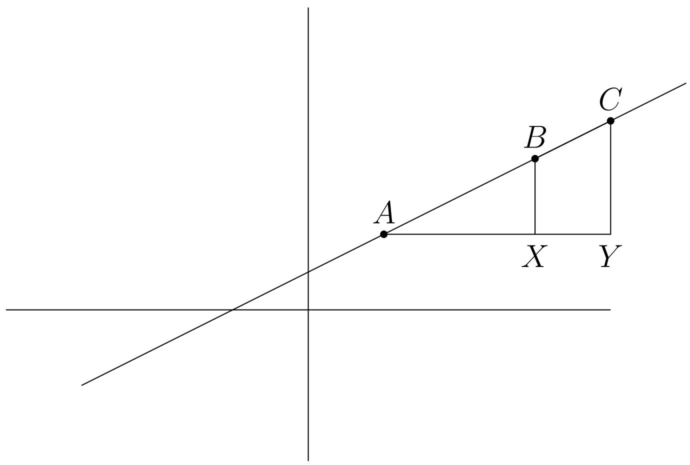
Theorem 2. *If two points have the same coordinates, then they are the same point. On the other hand, if two points have different coordinates, then they are different points.*

Proof. Let A and B be points in the xy plane with coordinates (a, r) and (b, s).
Suppose first that A and B have the same coordinates, so that $a = b$ and $r = s$. Then, based on Coordinate Theorem 1 proved above, we must have that AB is parallel to both the x-axis and the y-axis. But, the x and y axes are perpendicular, so obviously it's impossible for any line to be parallel to both, therefore AB must in fact not be a line at all! Based on Axiom 1, the only way for this to be possible is if A and B are the same point.
To prove that points that have different coordinates are different points, we can prove instead the contrapositive, namely that points that are the same have the same coordinates. But this is obvious - if two points are actually the same point, then their distances to any line are the same, therefore their distances to the x and y axes are the same. □

Theorem 3. *Every line not parallel to the x or y axes has a unique slope. This means that I can choose any right triangle whose hypotenuse is on the line and whose legs are parallel to the axes, and the slope will be the same number.*

Proof. Suppose I have a line l and points A, B, C on l, and suppose I draw right triangles $\triangle AXB$ and $\triangle AYC$ such that AB and AC are the hypotenuses, AX and AY are parallel to the x axis, and XB and YC are parallel to the y axis. Suppose I have a line l and points A, B, C on l, where the points have coordinates (x_a, y_a), (x_b, y_b), and (x_c, y_c). A right triangle with hypotenuse AB and legs parallel to the axes will have a y-parallel leg of length $y_b - y_a$ and an x-parallel leg of length $x_b - x_a$; similarly, a right triangle with hypotenuse AC and legs parallel to the axes will have its y and x legs of length $y_c - y_a$, $x_c - x_a$. Let these triangles be $\triangle AXB$ and $\triangle AYC$, where the legs parallel to the x axis are AX and AY, as shown in the figure below.
AX and AY must be the same line because they are parallel to each other (because they are both parallel to the x-axis) and contain a common point (A). We also know that B and C lie on the same line, because they are both on l. Therefore, $\angle XAB \cong \angle YAC$.
We thus know that $\triangle AXB \sim \triangle AYC$ by AA similarity because they are both right triangles

that share another common angle.
Because $\triangle AXB \sim \triangle AYC$, the ratios of corresponding sides are the same, thus $\frac{CY}{BX} = \frac{AY}{AX}$.
Rearranging, this gives $\frac{CY}{AY} = \frac{BX}{AX}$.
This proves that, in both triangles, the ratio of the length of the y leg to the x leg is the same. Thus, the slope of each triangle is the same.
It is not hard to see that, given any two triangles whose hypotenuse is on l and whose legs are parallel to the axes, one can perform the same argument several times as needed to prove that the slopes of the triangles are the same. Therefore, any line has only one slope. □



Theorem 4. *Given any number c, the graph of $x = c$ is a line parallel to the x-axis, and the graph of $y = c$ is a line parallel to the y-axis.*

Proof. Proof is left as homework. □

Theorem 5. *Given any nonzero number m, the graph of $y = mx$ is a line.*

Proof. Let g be the graph of $y = mx$, and let A, B, C be any three points in g, with coordinates (a, ma), (b, mb), (c, mc). Let X be the point at coordinates (b, ma), and let Y be the point at coordinates (c, ma). Then AX and AY are parallel to the x-axis by Coordinate Theorem 1; similarly, BX and CY are parallel to the y-axis by Coordinate Theorem 1. Therefore $AX \perp BX$ and $AY \perp BY$, thus AXB and AYC are right triangles.
Now, use Coordinate Theorem 1 to deduce the following line segment lengths: $AX = b - a$, $AY = c - a$, $BX = mb - ma$, $CY = mc - ma$. The same figure that we used in Coordinate Theorem 3 is also helpful here to see what is happening; if it helps, try drawing another diagram for this theorem, and label the coordinates of A, B, C, X, Y.
Now notice that $\frac{CY}{BX} = \frac{mc-ma}{mb-ma} = \frac{c-a}{b-a} = \frac{AY}{AX}$, and rearrange to get $\frac{CY}{AY} = \frac{BX}{AX}$.
By SAS triangle similarity, using $\angle AYC \cong \angle AXB$ and $\frac{CY}{AY} = \frac{BX}{AX}$, we have $\triangle AYC \sim \triangle ABX$. Therefore, $\angle XAB \cong \angle YAC$.
Notice that, by Axiom 3, $m\angle CAB = m\angle CAX - m\angle XAB$. Then, because X and Y are on the same ray from A, we can substitute Y for X in $\angle CAX$ to get $m\angle CAB = m\angle CAY - m\angle XAB$, which is equal to zero because the last two angles are congruent. Thus $m\angle CAB = 0°$ and therefore C, A, B must be on the same line.
The same argument can be used for any three points in g, thus all points in g must be on the same line, therefore g is itself a line. □

Theorem 6. *Given any two lines in the coordinate plane, if they are parallel lines, then their slopes are the same. On the other hand, if their slopes are the same, then they are parallel lines.*

Proof. Proof is left as homework. ☐

Theorem 7. *Given any two lines k and l in the coordinate plane, let their slopes be u and v. If $k \perp l$, then $uv = -1$. On the other hand, if $uv = -1$, then $k \perp l$.*

Proof. Proof is left as homework. ☐

Theorem 8. *The graph of the equation $x^2 + y^2 = r^2$ is a circle of radius r whose center is the origin. The graph of the equation $(x-a)^2 + (y-b)^2 = r^2$ is a circle of radius r whose center is the point with coordinates (a, b).*

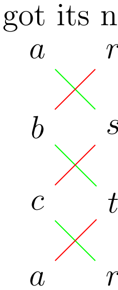
Proof. Based on what we discussed when we talked about the Pythagorean Theorem, we know that the distance from any point (x, y) to the origin is $\sqrt{x^2 + y^2}$. Therefore, if $x^2+y^2 = r^2$, then $\sqrt{x^2 + y^2} = r$, thus all points for which $x^2 + y^2 = r^2$ holds true must be a distance r from the origin. But the set of points a constant distance from a center point is a circle by definition, so the graph of this equation will be a circle of radius r whose center is the origin.
For the second case, one can similarly apply the formula for the distance between points to find that this equation describes a set of points that are a distance r from the point (a, b). ☐

Shoelace Theorem

The following is a curious theorem called the Shoelace Theorem, often also called the Shoelace Formula or Shoelace Equation, used for calculating the area of a triangle given the coordinates of its vertices. It actually has a more general form that can be used for polygons of more than three vertices, but I won't state that version here. Its proof takes some work and I won't state that here either, but it is useful to know the formula.

Here is the statement of the theorem, plus a diagram to hint at how it got its name.

Theorem 9 (Shoelace)**.** *Let $\triangle ABC$ be a triangle whose vertices have coordinates (a, r), (b, s), (c, t).*
Then the area of $\triangle ABC$ is equal to $\frac{1}{2}|as + bt + cr - br - cs - at|$.



Homework

1. Read pages 394-399 in the book and look over the examples. Note that the general equation of a line is $y = mx + b$, where m is the slope and b is the y-intercept. Y-intercept is the point where the line intersects the y-axis, the point (0,b).

2. What is the slope of a line whose equation is $y = 2x$? What is the slope of a line whose equation is $y = mx$?

3. In this problem you will find equations that describe some lines.
   (a) What is the equation whose graph is the y-axis?
   (b) What is the locus of points which are 5 units above the x-axis? Describe it as the graph of an equation.
   (c) Is the graph of $y = x$ a line? Which Coordinate Theorem proves or disproves this?
   (d) Find the equation of a line that contains the points $(1, -1)$, $(2, -2)$, and $(3, -3)$.

4. Prove Theorem 4.

5. A line l has slope $\frac{3}{5}$. What is the slope of a line parallel to l? What is the slope of a line perpendicular to l? (You may use any relevant Coordinate Theorems to explain your answer to this question.)

6. (a) Find k if $(1, 9)$ is on the graph of $y - 2x = k$.
   (b) Find k if $(1, k)$ is on the graph of $5x + 4y - 1 = 0$.

7. (a) Find the area of a triangle with vertices at $(-1, -2)$, $(4, -1)$, $(5, 4)$.
   (b) Find the area of a triangle with vertices at $(5, 4)$, $(0, 3)$, $(-1, -2)$.
   (c) Show that the quadrilateral with the vertices at $(-1, -2)$, $(4, -1)$, $(5, 4)$, $(0, 3)$ is a rhombus. Then, find its area.

8. (a) Given any numbers m, b, prove that the graph of $y = mx + b$ is a line.
   (b) Given any two points A, B, with coordinates (a, r), (b, s), find an equation whose graph is the line $\overleftrightarrow{AB}$.
   (c) Given any two points A, B, find an equation whose graph is the perpendicular bisector of AB.

9. Let l_1 be the graph of $y = x + 1$, l_2 be the graph of $y = x - 1$, m_1 be the graph of $y = -x + 1$, and m_2 be the graph of $y = -x - 1$.
   (a) Prove that l_1 and m_1 intersect at exactly one point. Label this point A and write down its coordinates.
   (b) Prove that l_2 and m_2 intersect at exactly one point. Label this point B and write down its coordinates.
   (c) Find the midpoint of AB and write down its coordinates.
   (d) Let C be the intersection point of l_1 with m_2, and D be the intersection point of l_2 with m_1. What kind of quadrilateral is $ABCD$?
   (e) Explain why l_1 and l_2 are parallel. What is the distance between them?

10. Prove Theorem 6.

***11.** Prove Theorem 7.