MATH 6: HOMEWORK 19

FEB 23, 2020

1. Triangle inequalities

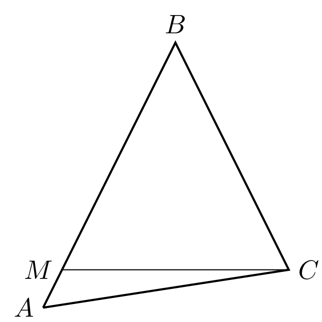
In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 12). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

Theorem 13. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 12, $m\angle A = m\angle C$ as base angles, which gives a contradiction.



Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$. On the other hand, by Exercise 7 of the previous homework, we have $m\angle BMC > m\angle A$, and by Axiom 3, we have $m\angle C = m\angle ACM + m\angle MCB > m\angle MCB$, so

$$m\angle C > m\angle MCB = m\angle BMC > m\angle A$$

so we have reached a contradiction.

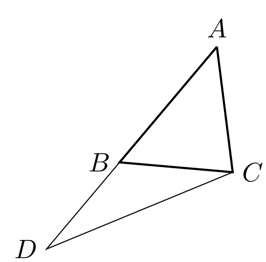
Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction. Therefore, the only possibility is that $BC > AB$. □

The converse of the previous theorem is also true: opposite a long side, there must be a big angle.

Theorem 14. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 15 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*



Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 13, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □

2. Special quadrilaterals

Among all quadrialterals, there are some that have special properties. In this section, we discuss three such types.

Definition 1. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

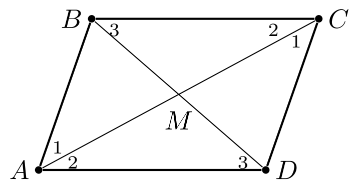
Theorem 16. *Let $ABCD$ be a rhombus. Then the diagonals are perpendicular, and the intersection point of diagonals is the midpoint for each of them.*

The proof of this theorem was one of the problems in your previous homework assignment.

Theorem 17. *Let $ABCD$ be a parallelogram. Then*

- $AB = CD$, $AC = BD$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By alternate interior angles theorem, angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ACB \cong \triangle CDA$. Therefore, $AB = CD$, $AC = BD$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.



Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by alternate interior angles theorem, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $CM = MD$. □

Theorem 18. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = CD$, $AC = BD$. Then $ABCD$ is a parallelogram.*

Homework

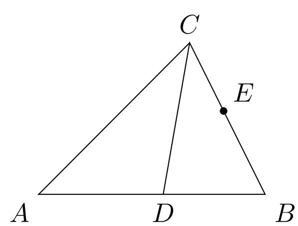
1. Do problem 19 on p 165, 13, 18, 19, 33 on pages 194 - 198 in the textbook.

2. Let $ABCD$ be a rectangle (i.e., all angles have measure 90°). Show that then, opposite sides are equal.

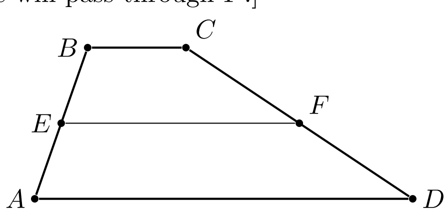
3. Prove Theorem 18

4. Show that each rhombus is a parallelogram.

5. Given $\triangle ABC$, let D be on AB such that $\angle ACD$ is equal to $\angle BCD$. Suppose we wish to place a point E on $\overline{BC}$ such that $\triangle CED$ is isosceles. Prove then that we must have $AC \parallel DE$.



6. In a trapezoid $ABCD$, with bases AD, BC, let E be the midpoint of side AB and F be the midpoint of side CD. Show that $EF \parallel AD$. [Hint: draw a line through E parallel to AD and show that this line will pass through F.]



***7.** Given the trapezoid from the previous problem, can you prove that $EF = \frac{1}{2}(AB + CD)$? The following figure may help: