MATH 10
ASSIGNMENT 22: SUBGROUPS

MAY 3, 2020

Definition. Let G be a group. A subgroup of G is a subset $H \subset G$ which is itself a group, with the same operation as in G. In other words, H must be

1. closed under multiplication: if $H_1, h_2 \in H$, then $h_1 h_2 \in H$
2. contain the group unit e
3. for any element $h \in H$, we have $h^{-1} \in H$.

Examples are given in problem 1 below.

Homework

1. Which of the following are subgroups?
(a) $G = \mathbb{Z}$ (with operation of addition), $H = 5\mathbb{Z}$ =multiples of 5.
(b) $G = \mathbb{Z}$ (with operation of addition), $H = \{n = 5k + 1\}$.
(c) G =all symmetries of regular n-gon, H=all rotations of regular n-gon
(d) G =all symmetries of regular n-gon, H=all reflections of regular n-gon

2. Let $\mathbb{Z}_n$ be the group of all remainders mod n, with operation of addition (it is commonly called the cyclic group of order n). Explain how one can identify this group with the group of all rotations of regular n-gon.

3. Let G be a group, and let $a \in G$. Consider the set of all powers of a:

$$H = \{a^n \mid n \in \mathbb{Z}\} \subset G$$

(note that n can be negative).
(a) Show that H is a subgroup (this is called the subgroup generated by a). Subgroups of this form are also called *cyclic* subgroups.
(b) Describe explicitly the cyclic subgroup in $\mathbb{Z}_{10}$ generated by 2; by 3; by 6.

4. Describe all subgroups in $\mathbb{Z}$ (hint: each such subgroup is cyclic).

5. Let $H \subset G$ be a subgroup. For any element $g \in G$, define the subset

$$[g] = gH = \{gh, h \in H\}$$

Subsets of this form are called *cosets*. Note that two different elements can define the same coset.
(a) List all cosets in the case when $G = \mathbb{Z}, H = 5\mathbb{Z}$.
(b) Show that two elements x, x' are in the same coset gH iff $x' = xh$ for some $h \in H$.
(c) Show that two cosets g_1H, g_2H either coincide (if $g_1 = g_2h$ for some $h \in H$) or do not intersect at all.