A and G 2. Classwork 9. November 17, 2019

Algebra.

Combinatorics.

Binomial coefficients.

Let us introduce the following notation:

$$_nC_m \equiv \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

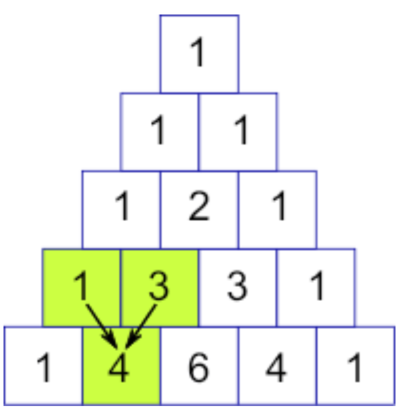
These numbers (pronounced "n Choose m") are called binomial coefficients. These numbers appear in many problems:

- The m-th entry in n-th line of the Pascal's triangle. Note that both n and m are counted from 0, not from 1: for example, $_2C_1 = 2$.
- The number of ways to choose m objects from the set of n different objects when the order is not important.
- The number of paths (going only right and up) from a square on the chessboard to the square m units up and k units to the right is $_{m+k}C_k$. For example, there $_{14}C_7$ paths from lower left to upper right of the usual 8×8 chessboard.
- The number of words of length exactly n which can be written using only 2 letters, say U and R, and which contain exactly m U's.

- and any more.. see problems below!

Problems:

1. Recall Pascal's triangle



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Compute the binomial coefficient $_5C_3$ and compare it with the 3rd entry of the 5th row.

2. Prove the following property of binomial coefficients:

$$\binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1}$$

   Explain the meaning of this formula for Pascal's triangle

3. The **Newton's binomial** is an expression representing the simplest n-th degree factorized polynomial of two variables, $P_n(x, y) = (x + y)^n$ in the form of the polynomial summation (i.e. expanding the brackets),

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{n-1} x\, y^{n-1} + \binom{n}{n} y^n$$

Using this formula expand the brackets in $(x + y)^5$, $(a - 2b)^4$. You can do it quickly if you use the Pascal's triangle.

4. To prove the Newton's binomial formula consider

$$(x + y)^n = (x + y)(x + y) \ldots (x + y)$$

When you expand the brackets in how many ways you can choose m x-s and (n-m) y-s from n brackets to obtain the coefficient in front of $x^m y^{n-m}$?

5. For an alternative prove of the Newton's binomial formula assume that you know coefficients of the expansion of $(x + y)^n$. Consider $(x + y)^{n+1}$ and write it as $(x + y)^n(x + y)$. If you expand this expression knowing coefficients for $(x + y)^n$ what coefficients will you obtain for $(x + y)^{n+1}$? Compare the found rule with the Pascal' triangle rule and with the result of the problem 2 above.

Let's go back to the problem number 3 and compare it with the following problem:

6. Expand brackets in the expression $2^n = (1 + 1)^n$ using Newton's binomial formula. What identity for binomial coefficients do you obtain? What does this identity mean for Pascal's triangle? Repeat the same exercise for $0 = (1 - 1)^n$.

7. Compute the number of paths (going only right and up) from the lower left to upper right squares of the usual 8×8 chessboard.

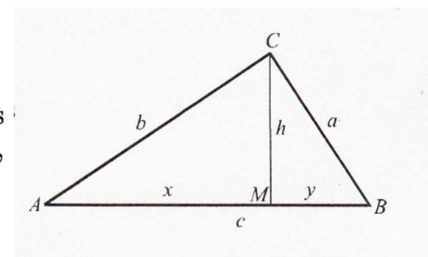
Geometry.

Pythagorean Theorem.

The following theorem is one of the oldest known to humanity. According to some studies, this result may have been known as early as 4 thousand years ago — long before Pythagoras or Euclid. There is a number of ways to prove this theorem. The proof given below is not the most geometrically intuitive — but it is the easiest one to derive from the axioms.

Theorem 28. *Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Denote $a = BC$, $b = AC$, $c = AB$. Then*

$$a^2 + b^2 = c^2$$



Proof. Let CM be the altitude from vertex C. By Theorem 27, triangles $\triangle ABC$, $\triangle ACM$, $\triangle CBM$ are similar; thus, $\frac{x}{b} = \frac{b}{c}$, so $x = b^2/c$. Similarly, $y = a^2/c$. Therefore,

$$c = x + y = \frac{a^2}{c} + \frac{b^2}{c}$$

Multiplying both sides by c, we get the Pythagorean theorem. □

This theorem can be reversed:

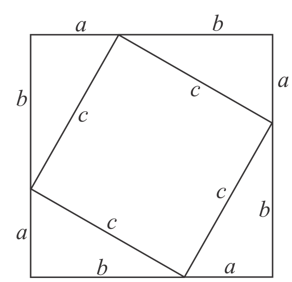
Theorem 29. *If $AB^2 = AC^2 + BC^2$, then $\triangle ABC$ is a right triangle, with right angle C.*

Proof. Construct a right triangle $\triangle A'B'C'$, with $m\angle C' = 90^\circ$ and $C'B' = CB$, $C'A' = CA$. Then, by Pythagorean theorem, $A'B' = \sqrt{AC^2 + AB^2} = AB$; thus, $\triangle A'B'C' \cong \triangle ABC$ by SSS. Therefore, $m\angle C = m\angle C' = 90^\circ$. □

Theorem 30 (Congruence by HL). *If two right triangles have equal hypothenuses and a pair of legs: $c = c'$, $a = a'$, then these triangles are congruent.*

Problems

8. Prove the Pythagorean Theorem (Theorem 28) using the following figure.



9. Prove Theorem 30.

10. In a trapezoid ABCD, with bases AD, BC, it is given that AD = 13, BC = 7, and the distance between bases is 4. It is also given that AB = CD. Find AB, AC.

11. Let ABC be a right triangle, with C the right angle, ∠A = 30° and AB = 1. Let D be a point on the side AB such that CD ⊥ AB. Can you find lengths DB and CD?

12. Find the length of the altitude AH in triangle ABC (see Figure). The side of each square cell is 1.