A and G 2. Class work 8.

Algebra.

General Vieta's formula.

It can be proven that for any general polynomial of degree n:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0\,, \quad a_n \neq 0$$

With the coefficient being real or complex numbers is known to have n (not necessarily distinct) (generally complex) roots $x_1,\ x_2, \dots ,\ x_n$

Vieta's formula relates the polynomial coefficients and roots as

$$\begin{cases} x_1 + x_2 + \cdots + x_{n-1} + x_n = -\dfrac{a_{n-1}}{a_n} \\ (x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + x_2x_4 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n = \dfrac{a_{n-2}}{a_n} \\ \vdots \\ x_1x_2 \cdot \ldots \cdot x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

Applied to quadratic equation;

$$P(x) = ax^2 + bx + c, \qquad x_1 + x_2 = -\frac{b}{a}, \quad x_1 \cdot x_2 = \frac{c}{a}$$

$$x_1 + x_2 = \frac{-b + \sqrt{D}}{2a} + \frac{-b - \sqrt{D}}{2a} = \frac{-b + \sqrt{D} - b - \sqrt{D}}{2a} = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{-b + \sqrt{D}}{2a} \cdot \frac{-b - \sqrt{D}}{2a} = \frac{\left(-b + \sqrt{D}\right) \cdot \left(-b - \sqrt{D}\right)}{4a^2} = \frac{(-b)^2 - \left(\sqrt{D}\right)^2}{4a^2}$$
$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Combinatorics.

Permutations.

There are many tasks when you need to count the number of possible outcomes. For example, there are 5 chairs and 5 kids in the room. In how many ways can kids sit on these chairs? The first kid can choose any chair. The second kid can choose any of the 4 remaining chairs, the third has a choice between the three chairs, and the fifth kid has no choice at all. Therefore, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways how all of them can choose their places. Thus obtained long expression, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, can be written as 5!. By definition:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ = 5! \qquad or \quad n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$$

This number 5! (or n!) shows the quantity of possible arrangement of 5 (n) objects, and is called permutations.

$$P = n!$$

Problems:

1. There are 10 books on the library shelf. How many different ways are there to place all these books on a shelf?

2. There are 10 books on the library shelf. 8 of them are authored by different authors and 2 are from the same author. How many different ways are there to place all these books on a shelf so that 2 books of one author will be next to each other?

   Because we want the two books of the same author be placed together, we can consider them as a single object and count the number of possible arrangements for 9 books, which is 9!. But in reality, for each of these arrangements, two books authored by the same author can be switch, so there are twice more possible arrangements, $2 \cdot 9!$.

   Now let's take a look on following problem:
   There are 20 desks in our class and only 9 students. How many different ways are there to sit in the math class?
   First student who came in the class has 20 desks to choose the place. Second student will have only 19 choices, and so on.
   The total number of possible ways to sit is
   $$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 = 20 \cdot (20-1) \cdot \ldots \cdot (20-9+1) = 60949324800$$
   In a general way we can ask about how many ways exists to arrange sample of m objects chosen out of n objects (order metter)? The first object can be chosen by n different ways, second one by $(n-1)$ different way, and so on. The total number of ways will be
   $$P(n,m) = n(n-1) \cdot \ldots \cdot (n-m+1)$$
   This is the number of permutations of m objects chosen from n.
   Can we write this formula in a shorter way?
   $$P(n,m) = n(n-1) \cdot \ldots \cdot (n-m+1) = \frac{n(n-1) \cdot \ldots \cdot (n-m+1) \cdot (n-m) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{(n-m) \cdot \ldots \cdot 3 \cdot 2 \cdot 1} = \frac{n!}{(n-m)!}$$
   If $m = n$, as in the first example, the formula is becoming
   $$P(n,n) = n = \frac{n!}{(n-n)!} = n!$$
   and it is clear that $0! = 1$, for everything to be consistent.

3. In how many different ways the first three places can be awarded, if 20 people participated in the competition? In this case the repetition is not allowed, same person can't be placed in first and second place.
   This problem does not allowed repetition of the same element, all kids are different persons.

4. Peter has 5 final exams, LA, Math, Science, Social Studies, and Art. He can get A, B, C, and D as grades. How many different ways are there for his report card to look like? In this case we have to arrange 4 different grade in groups of 5 exams. The result of the exam can be any grade, even the same as the grade he got fot another exam. For the first exam he can get any of the grade, so there is a choice of 4 grades for the first exam, as well as for any other exam.
   $$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$$
5. You have 4-digit lock and you forgot your code. You can check possible combinations with the speed 5 combination a minute. How long it will take you to open your locker?

Let's go back to the problem number 3 and compare it with the following problem:

6. How many different ways are there to create a command of 3 students out of 20 students of math class to take a participation in the math Olympiad. What is similar and what is different between these two problems?

The solution of the first one is

$$P(20,3)=\frac{20!}{(20-3)!}=20\cdot 19\cdot 18=6840$$

Does it matter, Alice is on the first place and Robert is on the second, or vice versa? Yes, it is a big difference for them. So, the group of three winners Alice, Robert, and Lia is different from the group Robert, Alice and Lia.

If we decide to solve the problem 6 the same way:

$$\frac{20!}{(20-3)!}=20\cdot 19\cdot 18$$

We definitely will get the group of students Alice, Robert, and Lia and Robert, Alice and Lia as different arrangements. Does it matter for the group of three students who is going to participate in the math Olympiad? Each group of three will be counted more times than needed. How many more times? Each group has $3\cdot 2\cdot 1=3!=6$ different ways to be arranged, so we have to divide our result by this:

$$C(20,3)=\frac{20!}{(20-3)!}:3!=\frac{20!}{(20-3)!\cdot 3!}=\frac{20\cdot 19\cdot 18}{6}=1140$$

This type of choosing groups of three out of 20 and order doesn't matter, is called combinations, $C(20,3)$. Also $20C3$, or $\binom{20}{3}$ notations can be used.

In a general way, if we want to choose set of m objects out on n objects, regardless of order

$$C(n,m)=\binom{n}{m}=\frac{n!}{m!\,(n-m)!}$$

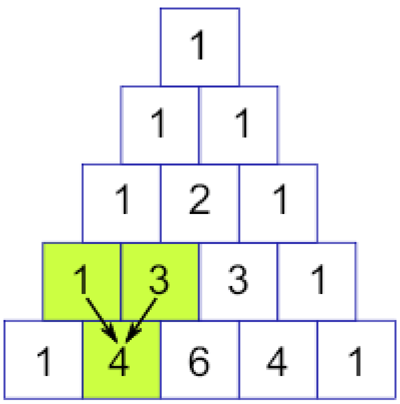
7. Prove the following:

$$\binom{n+1}{m+1}=\binom{n}{m}+\binom{n}{m+1}$$

The **Newton's binomial** is an expression representing the simplest n-th degree factorized polynomial of two variables, $P_n(x,y)=(x+y)^n$ in the form of the polynomial summation (i.e. expanding the brackets),

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y \ + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n-1}x\,y^{n-1} + \\ + \binom{n}{n}y^n$$

For $n \ = \ 1, 2, 3, \ldots$, these are familiar expressions,
$(x+y) \ = x+y$,
$(x+y)^2 = x^2 + 2xy + y^2$,
$(x+y)^3 = x^3 + 3x^2y \ + 3xy^2 + y^3$,
etc.
The Newton's binomial formula could be established either by directly expanding the brackets, or proven using the mathematical induction, but this is outside of our class curriculum (the method of mathematical induction is in level 9). The coefficients are forming the famos pascal's triangle:



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

An interesting consequence of the binomial theorem is obtained by setting both variables x and y equal to one. In this case, we know that $(1 \ + \ 1)^n \ = \ 2^n$, and so

$$\binom{n}{0} \ + \binom{n}{1} \ + \binom{n}{2} \ + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

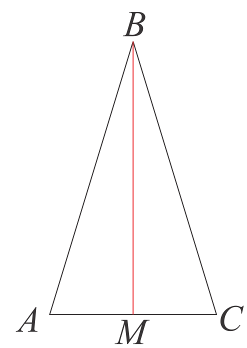
Geometry.

Geometry.

During the first part of the class we learned about statements, negation, how to construct complex statements with AND and OR logical operators. There is another complex logical statement which cab ne construct from two simple statements A and B: "If **A** then **B"**. If **A** is a true statement, then trustfulness of B will follow. We will discuss logic more next week, but now let's use it in geometrical theorems. Two weeks ago, we proved theorem "In isosceles triangle angles at the base are equal". There two statements here:
If the triangle is an isosceles triangle, then **its angles at the base are equal**. (1)
What about the revers position of these two statements?
If two angles of the triangle are equal, then **this triangle is an isosceles triangle**.
Does it necessarily follow from the theorem (1)? We already saw the example:

If two triangles are congruent, then all corresponding angles are equal, but the inverse predicate (this word also can be use for statement) is not true. From the statement "If corresponding angles of two triangles are equal", doesn't follow that two triangles are congruent.
Let's prove that

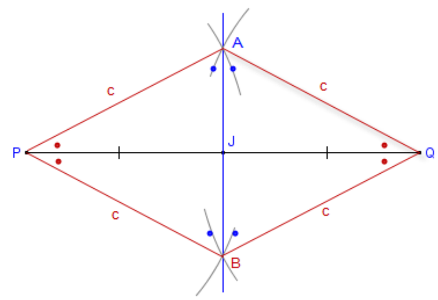
if two angles of a triangle are equal, triangle is an isosceles triangle.



In the triangle ABC $\angle BAC = \angle BCA$. Prove, that triangle ABC is an isosceles triangle. To prove this we need to prove that $|AB| = |BC|$ (or $[AB] \cong [BC]$). Let's draw the segment BM, a bisector of the angle $\angle ABC$

$\Delta ABM \cong \Delta MBC$, because angle $\angle A = \angle C, \angle ABM = \angle MBC$ (BM is a bisector), therefore $\angle BMA = \angle BMC$ and the side BM is a common side. Two triangles are congruent by the ASA test. Therefore $|AB| = |BC|$.

How to divide the segment by half with the compass and a straight edge?



	Argument	Reason
1	Line segments AP, AQ, PB, QB are all congruent	The four distances were all drawn with the same compass width c.
2	Triangles ΔAPQ and ΔBPQ are isosceles	Two sides are congruent (length c)
3	Angles AQJ, APJ are congruent	Base angles of isosceles triangles are congruent
4	Triangles ΔAPQ and ΔBPQ are congruent	Three sides congruent (sss). PQ is common to both.
5	Angles APJ, BPJ, AQJ, BQJ are congruent. (The four angles at P and Q with red dots)	Corresponding parts of congruent triangles are congruent
6	ΔAPB and ΔAQB are isosceles	Two sides are congruent (length c)
7	Angles QAJ, QBJ are congruent.	Base angles of isosceles triangles are congruent
8	Triangles ΔAPB and ΔAQB are congruent	Three sides congruent (sss). AB is common to both.
9	Angles PAJ, PBJ, QAJ, QBJ are congruent. (The four angles at A and B with blue dots)	Corresponding parts of congruent triangles are congruent

10	Triangles ΔAPJ, ΔBPJ, ΔAQJ, ΔBQJ are congruent	Two angles and included side (ASA)
11	The four angles at J - AJP, AJQ, BJP, BJQ are congruent	Corresponding parts of congruent triangles are congruent
12	Each of the four angles at J are 90°. Therefore AB is perpendicular to PQ	They are equal in measure and add to 360°
13	Line segments PJ and QJ are congruent. Therefore AB bisects PQ.	From (8), Corresponding parts of congruent triangles are congruent