MATH 8
ASSIGNMENT 8: LOGIC PROOFS CONTINUED

NOV 11, 2018

Basic logic operations

$\neg$: NOT (for example, NOT A): true if A is false, and false if A is true.
$\wedge$: AND (for example A AND B): true if both A, B are true, and false otherwise.
$\vee$: OR (for example A OR B): true if at least one of A, B is true, and false otherwise.
$\implies$: IF... THEN (for example $A \implies B$: "if A, then B): if A is false, automatically true; if A is true, it is true only when B is true

Proofs

A proof is a sequence of statements, startign with given ones and ending in a statement which we want to prove, and such that each statement in the sequence logically follows from the previous. In the simple case when all our statements can be written as combinations of the same elementary statements (which we can denote by letters $A, B, \dots$), using logical operations, it means

> For any combinations of values of letters $A, B, \dots$ which makes the previous statements true, the next statement is also true.

Here are some commonly used logic laws (all of them can be proved using truth tables):

- Given $A \implies B$ and A, we can conclude B (Modus Ponens)
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C.]
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ (De Morgan Law)
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ (law of contrapositive)

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

Common methods of proof

Conditional proof: To prove $A \implies B$, **assume** that A is true; derive B using this assumption.

Proof by contradiction: To prove A, assume that A is **false** and dervie a contradiction (i.e., something which is always false – e.g. $B \wedge \neg B$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

HOMEWORK

1. The following statement is sometimes written on highway trucks:

If you can't see my windows, I can't see you.

Can you write an equivalent statement without using word "not" (or its variations such as "can't").

2. Which of the statements below is equivalent to the statement If you do not work hard, you fail the course:

(a) If you work hard, you do not fail the course.
(b) If you failed the course, you did not work hard.
(c) If you did not fail the course, you worked hard.

3. Consider the follwoing statement:

You can't be happy unless you have a clear conscience.

Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter H for "you are happy" and C for "you have a clear conscience".

Note: proving this statement is not part of the assignment :).

4. Use proof by contradiction to prove the following statement:

If the square of an integer number n is even, then n itself is even.

You can use without proof that every integer number is either even (i.e., can be written in the form $n = 2k$, with integre k) or odd (i.e., can be written as $n = 2k + 1$, with integer k).

5. Prove that the equation $x^3 + 7x + 19 = 0$ has no positive real roots. You can use all the usual properties of real numbers, in particular properties of the order relation $<$.

6. Prove by contradiction that there does not exist a smallest positive real number.

7. Prove the following logical equivalence:

$$\neg (p \implies q) \iff (p \wedge \neg q)$$

8. (a) Prove the following logical equivalence:

$$p \vee (q \vee r) \iff (p \vee q) \vee r$$

Thus the logical or $\vee$ is said to be *associative*.

(b) Prove that the logical and $\wedge$ is also associative.

(c) **Disprove** the following logical equivalence:

$$(p \implies (q \implies r)) \iff ((p \implies q) \implies r)$$

[Hint: which of the statements is true if p, q, and r are all false?] Conclude that the logical implication $\implies$ is not associative.

9. Given logical statements m, p, q, let a denote the combined statement $(m \wedge p) \vee (\neg m \wedge q)$. In other words, $a \iff ((m \wedge p) \vee (\neg m \wedge q))$. Prove the following:

(a) If m is true, then $a \iff p$
(b) If m is false, then $a \iff q$

10. Asha, Benedict, and Cerys have bought a cake, but alas, due to bad planning, they are not sure if they all actually like the cake. To solve this issue in a terribly inefficient way, they decide to play a game: they find three lights, which we will call 1, 2, and 3, and if Asha likes the cake, Asha will switch lights 1 and 2; if Benedict likes the cake, Benedict will switch lights 2 and 3; if Cerys likes the cake, Cerys will switch lights 1 and 3. Initially all the lights are off (note that if a light that is off is switched, it turns on, and if a light that is on is switched, it turns off). Let us denote three logical statements: a will mean Asha likes the cake, b will mean Benedict likes the cake, and c will mean Cerys likes the cake. Thus, for example, if a, b, and c are all false, then all the lights remain off.

(a) If $a \iff b \iff c$, then which lights are on?

(b) If $(a \iff \neg b) \wedge (a \iff \neg c)$, then which lights are on?

(c) Now let l_1, l_2, and l_3 denote the logical statements that lights 1, 2, 3 are switched on, respectively. If $\neg((l_1 \wedge l_2) \implies l_3)$ and exactly one person likes the cake, then who likes the cake?

(d) If $\neg(l_1 \implies l_3) \vee \neg(l_2 \implies l_3)$ and exactly two people like the cake, then who likes the cake?

Partial Order Sets

Recall our first logic homework where we introduced a universe of linked objects, and denoted the linking by $<$. Any such linking relationship that satisfies the three rules given in the first homework is called a *partial order relation*, and any collection of objects linked by a partial order relation is called a *partial order set.*

1. Consider the two-dimensional plane $\mathbb{R}^2$, where points s, t are linked $s < t$ if t has both a larger x-coordinate and a larger y-coordinate than s.

(a) Check that this is a partial order relation, i.e. that it satisfies the three rules.

(b) Find an infinite collection of mutually linked points (i.e. an infinite collection of points such that any two distinct points s, t in the collection are linked as either $s < t$ or $t < s$).

(c) Find an infinite collection of mutually unlinked points (i.e., an infinite collection of points such that any two distinct points s, t in the collection are neither linked $s < t$ nor $t < s$).

(d) Find a point that is linked to an infinite collection of mutually unlinked points.

2. Consider the set of finite sequences of 0s and 1s. Let two sequences s, t be linked $s < t$ if s is a subsequence of t starting from the left. For example, $(0, 0, 1) < (0, 0, 1, 1, 0)$, but $(0, 0, 1)$ is not linked to $(0, 1, 0, 1)$ and $(0, 0, 1)$ is also not linked to $(1, 0, 0, 1, 0)$.

(a) Find an infinite collection of mutually linked sequences.

(b) Find an infinite collection mutually unlinked sequences.

(c) Find an object that's linked to an infinite collection of mutually unlinked sequences.