MATH 8
ASSIGNMENT 2: PASCAL TRIANGLE

SEP 23, 2018

Pascal triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

Every entry in this triangle is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by ${}_nC_k$. Note that both n and k are counted from 0, not from 1: for example, $\binom{2}{1} = 2$. Thus, these numbers are defined by these rules:

$$
\begin{aligned}
&{}_nC_0 = {}_nC_n = 1 \\
&{}_nC_k = {}_{n-1}C_{k-1} + {}_{n-1}C_k \qquad \text{for } 1 \le k \le n-1
\end{aligned} \tag{1}
$$

These numbers appear in many problems:

${}_nC_k =$ The number of paths on the chessboard going k units up and $n-k$ to the right

$=$ The number of words that can be written using k zeros and $n-k$ ones — see problem 4 below

$=$ **The number of ways to choose k items out of n (order doesn't matter) — see problem 5 below**

Problems

In this homework assignment (and in all other assignments in this class), many problems are non-trivial and require some thought. Try to start early. You are not expected to be able to solve all of the problems, so do not be discouraged if you can't solve some of them. The solutions are to be written on separate sheets of paper (as neatly as possible), with your name at the top, and handed back to me by the next class. Please make sure that you write not just the answer but also the solution, i.e. your reasoning showing how you arrived to this answer. Ideally, your solution should be such that someone who doesn't know how to solve this problem can read it and follow your arguments.

It is enough if you can write the answers in terms of factorials and binomial coefficients — it is not necessary to actually compute them: the answer like $\frac{13!}{3!}$ or ${}_{10}C_5$ is good enough.

1. If we want to choose a president, vice-president, and two assistants from a 15-member club, in how many wasy we can do it?

2. 5 kids come to a store to choose Halloween costumes. The store sells 25 different costumes. In how many ways can the kids choose the costumes? What if they want to choose so that all costumes are different?

3. Suppose I flip a coin three times, and I record its result each time (for example, the coin may land heads then tails then heads, which I will write as HTH, where order matters). I will refer to this three letter combination as the final result - for example, HHH is the only final result that has no tails.

(a) How many final results are there with exactly one tail?

(b) How many final results are there with exactly two tails?

4. Suppose that I now flip a coin n times, and want to find how many combinations there are in which I got tails exactly k times out of these n. Let us denote this number by $T(n, k)$.

(a) Show that $T(n, k) = a + b$, where a is the number of combinations where the first toss was heads, and b is the number of combinations where the first toss was tails.

(b) Show that $a = T(n-1, k)$. What about b?

(c) Deduce from the above that $T(n, k) = {}_nC_k$.

(d) If I toss a coin 8 times, what is the probablitiy that exactly 4 of them are tails?

5. Suppose I have a group with n members in it and from them I must choose an executive committee of k members. Let $M(n, k)$ refer to the number of ways there are to choose this committee. Unfortunately, there is a problematic individual in the group, who we will refer to as X. Let a denote the number of committees I could form excluding X, and b denote the number of committees I could form including X.

(a) Prove that $a + b = M(n, k)$.

(b) Prove that $a = M(n-1, k)$, and $b = M(n-1, k-1)$.

(c) Deduce from the above that $M(n, k) = {}_nC_k$, i.e.

Number of way to choose k persons out of n (order doesn't matter) $= {}_nC_k$

6. (a) An artist has 5 paintings. He needs to choose 3 paintings to include in an art show. How many ways are there of doing this?

(b) The same artist now needs to choose 3 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).

(c) What if he needs to select 4 paintings? Do both versions: for the show (oreder doesn't matter) and for the catalog (order matters).

7. Five octopi are working at the beachs local landfood restaurant. They want to assign lunch shifts, so that some of the octopi can have lunch from 12:00pm to 1:00pm, and the others can have lunch from 1:00pm to 2:00pm.

(a) If they decide to have two octopi take the first shift and three take the second, how many possible ways are there to assign shifts?

(b) What it they want to have it the other way around, with three octopi taking the first shift and two taking the second?

8. What is the sum of all numbers in n-th row of Pascal triangle? Can you guess the pattern — and once you guessed it, justify your guess.

***9.** Are there any rows in the Pascal triangle where all numbers are odd? Which rows are they?