MATH 8: ASSIGNMENT 17

FEB 10, 2019

Circles

Given a circle λ with center O,

- A radius is any line segment from O to a point A on λ,
- A chord is any line segment between distinct points A, B on λ,
- A diameter is a chord that passes through O,
- A tangent line is a line that intersects the circle exactly once; if the intersection point is A, the tangent is said to be the tangent through A.

Moreover, we say that two circles are tangent if they intersect at exactly one point.

Theorem 20. *Let A be a point on circle λ centered at O, and m a line through A. Then m is tangent to λ if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to λ at A.*

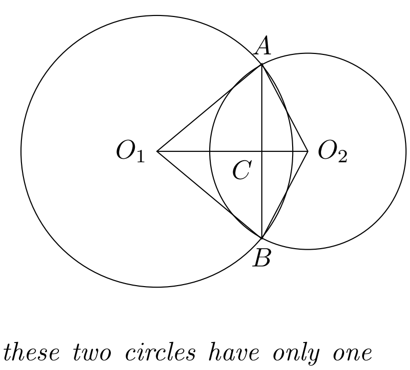
Proof. First we prove (m is tangent to λ) $\implies$ ($m \perp \overline{OA}$). Suppose m is tangent to λ at A but not perpendicular to $\overline{OA}$. Let $\overline{OB}$ be the perpendicular to m through O, with B on m. Construct point C on m such that $BA = BC$; then we have that $\triangle OBA \cong \triangle OBC$ by SAS, using $OB = OB$, $\angle OBA = \angle OBC = 90°$, and $BA = BC$. Therefore $OC = OA$ and hence C is on λ. But this means that m intersects λ at two points, which is a contradiction.
Now we prove ($m \perp \overline{OA}$) $\implies$ (m is tangent to λ). Suppose m passes through A on λ such that $m \perp \overline{OA}$. If m also passed through B on λ, then $\triangle AOB$ would be an isosceles triangle since $\overline{AO}$, $\overline{BO}$ are radii of λ. Therefore $\angle ABO = \angle BAO = 90°$, i.e. $\triangle AOB$ is a triangle with two right angles, which is a contradiction. □

Notice that, given point O and line m, the perpendicular $\overline{OA}$ from O to m (with A on m) is the shortest distance from O to m, therefore the locus of points of distance exactly OA from O should line entirely on one side of m. This is essentially the idea of the above proof.

Theorem 21. *Let $\overline{AB}$ be a chord of circle λ with center O. Then O lies on the perpendicular bisector of $\overline{AB}$. Moreover, if C is on $\overline{AB}$, then C bisects $\overline{AB}$ if and only if $\overline{OC} \perp \overline{AB}$.*

Proof. Let m be the perpendicular bisector of $\overline{AB}$. The center O of λ is equidistant from A, B by the definition of a circle, therefore by Theorem 14, O must be on m. Let m intersect $\overline{AB}$ at D. We then have that D is the midpoint of $\overline{AB}$ and also the foot of the perpendicular from O to $\overline{AB}$.
Then if C bisects $\overline{AB}$, C lies on the perpendicular bisector m of $\overline{AB}$, which passes through O, thus $\overline{OC} \perp \overline{AB}$. Lastly if $\overline{OC} \perp \overline{AB}$, then because there is only one perpendicular to $\overline{AB}$ through O, we must have $C = D$ and hence C is the midpoint of $\overline{AB}$. □

Theorem 22. *Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B. Then $\overline{AB} \perp \overline{O_1O_2}$.*



Proof. Let l be the perpendicular bisector of AB. By the previous theorem, l contains both centers: $O_1 \in l$, $O_2 \in l$. Thus, $l = \overline{O_1O_2}$, so $\overline{O_1O_2}$ is the perpendicaulr bisector of AB; in particular, they are perpendicular. □

Theorem 23. *Let*
omega$_1$, ω_2 be circles that are both tangent to line m at point A. Then these two circles have only one common point, A. Such circles are called tangent.

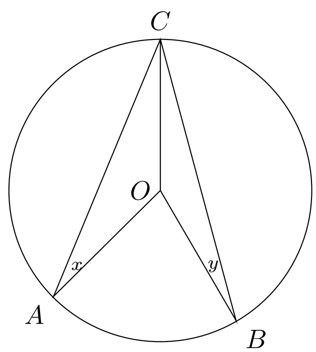
Proof. By Theorem 20, radiuses O_1A and O_2A are both perpendicular to m at A; since there can only be one perpendicular line to m at given point, it means that O_1, O_2, and A are on the same line, and that m is perpendicular to O_1O_2 at A.

Now, suppose, by contradiction, that ω_1, ω_2 intersect at point $B \neq A$. Then by the previous theorem, $\overline{AB} \perp \overline{O_1O_2}$, therefore both $\overline{AB}$ and m are perpendicular to $\overline{O_1O_2}$ through A. We must therefore have that B is on m, but m is tangent to ω_1 through A, thus has only one intersection with ω_1, which is a contradiction. □

Arcs and Angles

Consider a circle λ with center O, and an angle formed by two rays from O. Then these two rays intersect the circle at points A, B, and the portion of the circle contained inside this angle is called the arc subtended by $\angle AOB$.

Theorem 24. *Let A, B, C be on circle λ with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ.*



Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.

Case 1. Draw in segment $\overline{OC}$. Denote $m\angle A = x$, $m\angle B = y$. Since $\triangle AOC$ is isosceles, $m\angle AC) = x$; similarly $m\angle BCO = y$, so $m\angle ACB = x + y$, and $m\angle AOC = 180° - 2x$, $m\angle BOC = 180° - 2y$. Therefore, $m\angle AOC + m\angle BOC = 360° - 2(x + y)$. This implies $m\angle AOB = 2(x + y)$. □

As a result of Theorem 24, we get that any triangle $\triangle ABC$ on λ where $\overline{AB}$ is a diameter must be a right triangle, since the angle $\angle ACB$ has half the measure of angle $\angle AOB$, which is 180°.
The idea captured by the concept of an arc and Theorem 24 is that there is a fundamental relationship between angles and arcs of circles, and that the angle 360° can be thought of as a full circle around a point.

Homework

1. Prove that, given a segment $\overline{AB}$, there is a unique circle with diameter $\overline{AB}$.
2. Given lines $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ such that $\overline{AD}$, $\overline{BC}$ intersect at E and $AE = ED$, prove that $BE = EC$.
3. Prove that if a diameter of circle λ is a radius of circle ω, then λ, ω are tangent.
4. Complete the proof of Theorem 24 by proving the cases where O is not inside the angle $\angle ACB$. [Hint: for one of the cases, you may need to write $\angle ACB$ as the difference of two angles.]
5. Prove the converse of Theorem 24: namely, if λ is a circle centered at O and A, B, are on λ, and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on λ. [Hint: we need to prove that $OC = OA$; consider using a proof by contradiction, using Theorem 12.]
6. Let $\overline{AB}$, $\overline{CD}$ both have midpoint E. Prove that $ACBD$ is a parallelogram.
7. Given points A, B, C such that $AB = AC$, complete a straightedge-compass construction of a rhombus $ABDC$.
8. Prove that, given two distinct points A, B on circle λ which are on the same side of diameter $\overline{CD}$ of λ, that $CB \neq CA$.
9. Given triangle $\triangle ABC$, complete a straightedge-compass construction of a circle that passes through A, B, C. Deduce that given any three points A, B, C that form a triangle (i.e. are not on the same line), there exists a unique circle through these points.
10. Let $\overline{AB}$, $\overline{CD}$ both have midpoint E and let F, G be points such that $BECF$ and $AEDG$ are parallelograms. Prove that E is the midpoint of FG.