FACTORING QUADRATIC EQUATIONS

Review: **Roots. Factors. Viète's formulas**

A polynomial function of degree n is a function $x \to y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where n is a nonnegative integer and $a_n \neq 0$ **Definition:** a is a ***root of a polynomial function*** $P(x)$ if $P(a) = 0$.
Definition: $x - a$ is a factor of a polynomial function $P(x)$ if we can write $P(x) = Q(x) \cdot (x - a)$ for some non-zero polynomial $Q(x)$ with a degree $deg(Q(x)) = deg(P(x)) - 1$.
! *Too many things are called just "polynomials"* There are some distinctions to be made:

- $x \to y = f(x) = ax^2 + bx + c$ is a polynomial function of second degree : when we define a function we identify a domain where x takes values. In general we think of this polynomial function for x taking real number values, but we can define it to take only integer values and then their graphs will be very different.
- $ax^2 + bx + c = 0$ is its associated quadratic(or 2-nd degree) polynomial equation.

The Factor Theorem: $x - a$ is a factor of a polynomial function $P(x) \iff x = a$ is a root of the polynomial function $P(x) \iff x = a$ is an x-intercept of the graph of polynomial function $P(x)$
From the Factor Theorem, if x_1 and x_2 are roots of the second degree polynomial function $f(x) = ax^2 + bx + c$, then $f(x) = a(x - x_1)(x - x_2) = a(x^2 - x_1 x - x_2 x + x_1 x_2) = a(x^2 - (x_1 + x_2)x + x_1 x_2)$. Thus, if $a = 1$, then

$$x_1 + x_2 = -b \text{ and } x_1 x_2 = c$$

.

These formulas can be used to guess factors and find roots.

Solve

- Factor using Viète's formulas the following second degree polynomials. Check your results by polynomial division.
  1. $x^2 + 9x + 14$
  2. $8x^2 - 24x + 16$
  3. $z^4 + z^2 - 12$
- Find the solutions of the equation $x^2 = 8$
- Find the solutions of the equation $x^2 - D = 0$
- Find the solutions of the equation $(x - 4)^2 - 64 = 0$
- Find the solutions of the equation $(x - 4)^2 - D = 0$

Completing the square

The "completing the square" method uses the formula

$$(x + y)^2 = x^2 + 2xy + y^2$$

and forces a "square" out of the terms in x^2 and x, and tries to rewrite the quadratic polynomial as $a(x - h)^2 + k$
For instance, we can rewrite: $x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 7 = (x + 3)^2 - 7$, and then, $x^2 + 6x + 2 = 0$ if and only if $(x + 3)^2 = 7$, which gives $x + 3 = \sqrt{7}$ or $x + 3 = -\sqrt{7}$. Thus the roots are $x_1 = -3 - \sqrt{7}$ and $x_2 = -3 + \sqrt{7}$.
So, more generally but supposing $a = 1$, we can write

$$x^2 + bx + c = x^2 + 2 \cdot \frac{b}{2} \cdot x + c = x^2 + 2 \cdot \frac{b}{2} \cdot x + \frac{b^2}{2^2} - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{D}{4}, \text{ where } D = b^2 - 4c.$$

Now, solving $x^2 + bx + c = 0$ becomes equivalent to solving

$$\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$$

For the general case (a is not 1, nor zero), we can first divide everything by a, to equivalently solve

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Longleftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}\text{ , and now } D = b^2 - 4ac$$

D is generally called the discriminant of the 2-nd degree equation because it is used to distinguish how many real number solutions the equation has.
In order to have solutions, we need $D \geq 0$. Otherwise, if $D < 0$, there are no real number solutions. So, when $D \geq 0$, we have a direct way of obtaining the solutions, called the discriminant formula

$$x + \frac{b}{2a} = \pm\frac{\sqrt{D}}{2a} \Longleftrightarrow x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

Solve

- Complete the square for $x^2 + 12x - 5$ in the form $(x - B)^2 - D$
- Complete the square for $x^2 - 5x + 5 = 0$ in the form $(x - B)^2 - D$

End behavior of polynomial functions

End behavior of a polynomial. Axes of symmetry

We have two cases x becomes larger, and larger in the positive direction, denoted by $x \to \infty$, and x becomes larger, and larger in the negative direction, denoted by $x \to -\infty$

Examples

- x^2, $-x^2$, x^4, $-x^4$ (identify points/axes of symmetry)
- x^3, $-x^3$, x^5, $-x^5$ (identify points/axes of symmetry)

In general, the end behavior is determined by the term that contains the highest power of x, called the leading term. Why? Because when x is large, all the other terms are small compared to the leading term.

Solve

1. Solve the following quadratic equations by both completing to the square and the discriminant formula. Check your results by using the Viète's formulas.
   (a) $x^2 - 5x + 5 = 0$
   (b) $\dfrac{x}{x-2} = x - 1$
   (c) $x^2 = 1 + x$
   (d) $2x(3 - x) = 1$
2. For what values of b does the quadratic polynomial function defined on real numbers $x \to x^2 + bx + 14$ have no real roots? Exactly one root? Two distinct real roots?
3. If $x + \frac{1}{x} = 7$. Find $x^2 + \frac{1}{x^2}$, and $x^3 + \frac{1}{x^3}$
4. Think how should Viète's formulas look for
$$ax^3 + bx^2 + cx + d = 0$$
5. Given the polynomial functions
   (a) $f(x) = (x - 1)(x + 2)(x - 3)$,
   (b) $f(x) = (x - 4)(x + 5)$
   (c) $f(x) = (x - 4)^2$

   express the function as a polynomial in general form and determine the leading term, degree, and its end behavior.