Truth Tables, Tautologies, and Logical Equivalences

- Mathematicians normally use a two-valued logic: Every statement is either True or False.
- A statement in sentential logic is built from simple statements using the logical connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. The truth or falsity of a statement built with these connectives depends on the truth or falsity of its components.
- The compound statement $P \rightarrow (Q \vee \sim R)$ is built using the logical connectives $\rightarrow$, $\vee$, and $\sim$. The truth or falsity of $P \rightarrow (Q \vee \sim R)$ depends on the truth or falsity of P, Q, and R.
- A truth table shows how the truth or falsity of a compound statement depends on the truth or falsity of the simple statements from which it's constructed. So we'll start by looking at truth tables for the five logical connectives.

<table>
<tr><th>And</th><th>Or</th><th>If… then…</th><th>If and only if…</th></tr>
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<td><table><tr><th>P</th><th>Q</th><th>$P \wedge Q$</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>F</td></tr></table></td>
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<td>$P \wedge Q$ should be true when both P and Q are true, and false otherwise.</td>
<td>$P \vee Q$ is true if either P is true or Q is true (or both — remember that we're using "or" in the inclusive sense). It's only false if both P and Q are false.</td>
<td>$P \rightarrow Q$ is only false when the hypothesis is true and the conclusion is false.</td>
<td>$P \leftrightarrow Q$ means that P and Q are equivalent. So the double implication is true if P and Q are both true or if P and Q are both false; otherwise, the double implication is false.</td>
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- A tautology is a formula which is "always true" — that is, it is true for every assignment of truth values to its simple components. You can think of a tautology as a rule of logic.
- The opposite of a tautology is a contradiction, a formula which is "always false". In other words, a contradiction is false for every assignment of truth values to its simple components.

Practice:

1) Construct a truth table for the formula $\sim P \wedge (P \rightarrow Q)$.
2) Show that $(P \rightarrow Q) \vee (Q \rightarrow P)$ is a tautology.
3) Construct a truth table for $(P \rightarrow Q) \wedge (Q \rightarrow R)$.
4) Show that $P \rightarrow Q$ and $\sim P \vee Q$ are logically equivalent.

ANSWERS

1) Construct a truth table for the formula $\sim P \wedge (P \rightarrow Q)$.

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \wedge (P \rightarrow Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

2) Show that $(P \rightarrow Q) \vee (Q \rightarrow P)$ is a tautology.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

3) Construct a truth table for $(P \rightarrow Q) \wedge (Q \rightarrow R)$.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

4) Show that $P \rightarrow Q$ and $\sim P \vee Q$ are logically equivalent.

P	Q	$P \rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $P \rightarrow Q$ and $\sim P \vee Q$ are identical, the two statements are logically equivalent.

Inverse, Converse, and Contrapositive:

Conditional Statements have two parts:

The **hypothesis** is the part of a conditional statement that follows **"if"** (when written in if-then form.) It is the given information, or the condition.

The **conclusion** is the part of an if-then statement that follows **"then"** (when written in if-then form.) It is the result of the given information.

Conditional statements can be written in **"if-then"** form to emphasize which part is the hypothesis and which is the conclusion.

Writing Conditional Statements

Hint: Turn the subject into the hypothesis.

Vertical angles are congruent. *can be written as...* If two angles are vertical, then they are congruent.

Seals swim. *can be written as...* If an animal is a seal, then it swims.

Inverse	negate the hypothesis and conclusion of the original conditional	~p → ~q	Right angle is defined as- an angle whose measure is 90 degrees. If you are to write it as inverse statement, it can be done like: If an angle is not a right angle, then it does not measure 90.
Converse	interchange the hypothesis and the conclusion	q → p	If you are to write the converse of: "If two lines don't intersect, then they are parallel", it can be written as "If two lines are parallel, then they don't intersect."
Contrapositive has the same truth value as the original statement.	negate both the hypothesis and conclusion and also interchange these negations	~ q → ~p	The statement 'A triangle is a three- sided polygon' is true; its contrapositive, 'A polygon with greater or less than three sides is not a triangle' is true too.

Practice:

1) Write the converse, inverse, and contrapositive of "If you are a brunette, then you have brown hair."
2) Write the converse, inverse, and contrapositive of "If you are a guitar player, then you are a musician. "

ANSWERS:

1) Converse: If you have brown hair, then you are a brunette.

   Inverse: If you are not a brunette, then you do not have brown hair.

   Contrapositive: If you do not have brown hair, then you are not a brunette.

2) Converse: If you are a musician, then you are a guitar player.

   Inverse: If you are not a guitar player, then you are not a musician.

   Contrapositve: If you are not a musician, then you are not a guitar player.

Tautologies, Logical Equivalences, Laws of Inference

DeMorgan'sLaw $\sim(P \vee Q) \leftrightarrow (\sim P \wedge \sim Q)$

DeMorgan'sLaw $\sim(P \wedge Q) \leftrightarrow (\sim P \vee \sim Q)$

Contrapositive $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$

Modus Ponens	Modus Tollens	hypothetical syllogism
$p \implies q$ p $\therefore q$	$p \implies q$ $\sim q$ $\therefore \sim p$	$p \implies q$ $q \implies r$ $\therefore p \implies r$
Addition	**Disjunctive syllogism**	**simplification**
p $\therefore p \vee q$	p ∨ q ~p ∴q	p ^ q $\therefore p$ $\therefore q$

Practice

1) Given: $p \wedge (p \to q)$ Prove: *q*

2) With these hypotheses:
"It is not sunny this afternoon and it is colder than yesterday."
"We will go swimming only if it is sunny."
"If we do not go swimming, then we will take a canoe trip."
"If we take a canoe trip, then we will be home by sunset."

   Using the inference rules, construct a valid argument for the conclusion:
"We will be home by sunset."

ANSWERS

1)

Statement	Reason
$p \wedge (p \rightarrow q)$	Given
P	Simplification
p → q	Simplification
q	Modus Ponens

2) Choose propositional variables:

p : "It is sunny this afternoon." *r* : "We will go swimming." *t :* "We will be home by sunset."

q : "It is colder than yesterday." *s :* "We will take a canoe trip."

Translation into propositional logic:

Hypotheses: $\neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$

Conclusion: t

Statement	Reason
~p ^ q	Given
~p	Simplification
q	Simplification
r → p	Given
~r	Modus Tollens
~r → s	Given
S	Modus Ponens
s → t	Given
t	Modus Ponens