AM1. Class work 25.

Algebra.

There are many tasks when you need to count the number of possible outcomes. For example, there are 5 chairs and 5 kids in the room. In how many ways can kids sit on these chairs? The first kid can choose any chair. The second kid can choose any of the 4 remaining chairs, the third has a choice between the three chairs, and the fifth kid has no choice at all. Therefore, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways how all of them can choose their places. Thus obtained long expression, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ , can be written as 5!. By definition:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ = 5! \qquad or \quad n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$$

This number 5! (or n!) shows the quantity of possible arrangement of 5 (n) objects, and is called permutations.

$$P = n!$$

**Problems**:

1. There are 10 books on the library shelf. How many different ways are there to place all these books on a shelf?

2. There are 10 books on the library shelf. 8 of them are authored by different authors and 2 are from the same author. How many different ways are there to place all these books on a shelf so that 2 books of one author will be next to each other?

Because we want the two books of the same author be placed together, we can consider them as a single object and count the number of possible arrangements for 9 books, which is 9!. But in reality, for each of these arrangements, two books authored by the same author can be switch, so there are twice more possible arrangements, $2 \cdot 9!$.

Now let’s take a look on following problem:
There are 20 desks in our class and only 9 students. How many different ways are there to sit in the math class? How long it will take to try all of them, if you need 1 minute to switch places. First student who came in the class has 20 desks to choose the place. Second student will have only 19 choices, and so on.
The total number of possible ways to sit is

$$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 = 20 \cdot (20-1) \cdot \ldots \cdot (20-9+1) = 60949324800$$

It will take 42325920 days (or almost 116000 years) to try all of them.
In a general way we can ask about how many ways exists to arrange sample of m objects chosen out of n objects? The first object can be chosen by $n$ different ways, second one by $(n-1)$ different way, and so on. The total number of ways will be

$$P(n, m) = n(n-1) \cdot \ldots \cdot (n-m+1)$$

This is the number of permutations of $m$ objects chosen from $n$.
Can we write this formula in a shorter way?

$$P(n,m) = n(n-1)\cdot\ldots\cdot(n-m+1) = \frac{n(n-1)\cdot\ldots\cdot(n-m+1)\cdot(n-m)\cdot\ldots\cdot 3\cdot 2\cdot 1}{(n-m)\cdot\ldots\cdot 3\cdot 2\cdot 1} = \frac{n!}{(n-m)!}$$

If $m = n$, as in the first example, the formula is becoming

$$P(n,n) = n = \frac{n!}{(n-n)!} = n!$$

and it is clear that $0! = 1$, for everything to be consistent.

3. In how many different ways the first three places can be awarded, if 20 people participated in the competition? In this case the repetition is not allowed, same person can't be placed in first and second place.
   This problem does not allowed repetition of the same element, all kids are different persons.

4. Peter has 5 final exams, LA, Math, Science, Social Studies, and Art. He can get A, B, C, and D as grades. How many different ways are there for his report card to look like? In this case we have to arrange 4 different grade in groups of 5 exams. The result of the exam can be any grade, even the same as the grade he got fot another exam. For the first exam he can get any of the grade, so there is a choice of 4 grades for the first exam, as well as for any other exam.
   $$4\cdot 4\cdot 4\cdot 4\cdot 4 = 4^5$$
5. You have 4-digit lock and you forgot your code. You can check possible combinations with the speed 5 combination a minute. How long it will take you to open your locker?

   Let's go back to the problem number 3 and compare it with the following problem:

6. How many different ways are there to create a command of 3 students out of 20 students of math class to take a participation in the math Olympiad. What is similar and what is different between these two problems?

   The solution of the first one is

   $$P(20,3) = \frac{20!}{(20-3)!} = 20\cdot 19\cdot 18 = 6840$$

   Does it matter, Alice is on the first place and Robert is on the second, or vice versa? Yes, it is a big difference for them. So, the group of three winners Alice, Robert, and Lia is different from the group Robert, Alice and Lia.

   If we decide to solve the problem 6 the same way:

   $$\frac{20!}{(20-3)!} = 20\cdot 19\cdot 18$$

   We definitely will get the group of students Alice, Robert, and Lia and Robert, Alice and Lia as different arrangements. Does it matter for the group of three students who is going to participate in the math Olympiad? Each group of three will be counted more times than needed. How many more times? Each group has $3\cdot 2\cdot 1 = 3! = 6$ different ways to be arranged, so we have to divide our result by this:

$$C(20,3) = \frac{20!}{(20-3)!} : 3! = \frac{20!}{(20-3)! \cdot 3!} = \frac{20 \cdot 19 \cdot 18}{6} = 1140$$

This type of choosing groups of three out of 20 and order doesn't matter, is called combinations, $C(20,3)$. Also $20C3$, or $\binom{20}{3}$ notations can be used.

In a general way, if we want to choose set of $m$ objects out on $n$ objects, regardless of order

$$C(n,m) = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

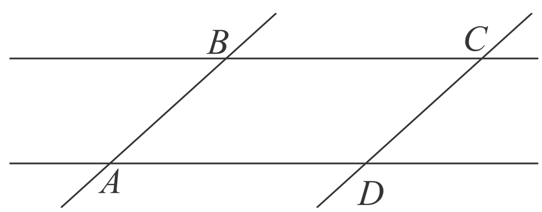
More problems:

7. I have 7 cookies in total, 2 oatmeal cookies, 3 chocolate chips cookies, and 2 sugar cookies. How many different ways are there to eat this all cookies?
8. Mother has 2 apples and 3 pears. Each day she gives one fruit to her kid for lunch. How many different orders are there to give these fruits?
9. How many different ways are there to color the table 2x2 in white and black color? 3x5?
10. There are 3 starters, 4 entrees, and 4 desserts in the price fix dinner. How many different ways are there to fix your diner?

**Geometry**.

1. What is longer, the altitude or the median drawn to the same side of the triangle?
2. In the isosceles triangle sides are 8 and 10 cm. Find the length of the altitude drawn to the base.

**Parallelogram** is a simple (non-self-intersecting) quadrilateral with two pairs of parallel sides.

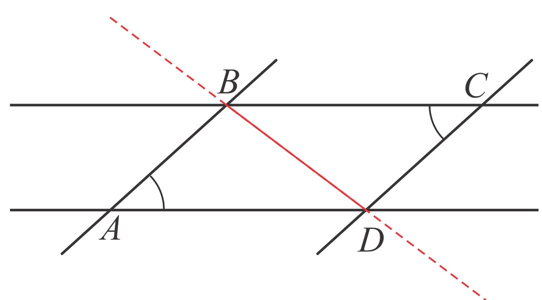



What else can we say about parallelogram? Is this enough as a definition?

Parallel sides of a parallelogram are equal. Does this statement need to be included into the definition or it can be proved based on the fact that they are parallel?

Let $ABCD$ be a parallelogram, we need to prove that $|AB| = |CD|$ and $|AD| = |BC|$.

Segment BD is a diagonal of the parallelogram. Angle $\angle ABD = \angle BDC$ as alternate interior angles (we can prove it based on the 5$^{th}$ Euclid postulate), as well as angles $\angle BDA = \angle DBC$. BD is a common side for the triangles. Therefore, triangles ABD and BDC are congruent, based on the ASA criteria and have equal corresponding sides. Conclusion: $|AB| = |CD|$ and $|AD| = |BC|$.




Opposite angles of a parallelogram are equal, and adjacent angles are supplementary.

Diagonals of a parallelogram intersect at midpoint of both segments. Prove it. Also prove the converse theorem – if two diagonals of a quadrilateral intersect at midpoint of both, then this quadrilateral is a parallelogram.