

Algebra.

A speech of a person or a text written on paper contain sentences. This is the way how we exchange the information between us. The information in every sentence can be a true fact, false, or sometime we just can't say is it true or false. For example, the sentence:

“The Earth is rotating around the Sun” is true.

“Paris is the capital of Germany” is false.

“Math is fun!” or “What time is it?” are the sentences we can't tell either it's true or false. Can you tell which sentence is “true”, “false”, or we can't tell:

- a. “22 is an even number”
- b. “44 is an odd number”
- c. “1001 is a cool number!”

Let's define “a statement” as a sentence about which we can tell (sometime after difficult process of proving) either it is true or false. For example, our first sentence “The Earth is rotating around the Sun” was proved to be true after hundreds of years of discussions. The second sentence, “Paris is the capital of Germany”, can be proved wrong after we will check it in the dictionary (assuming that we never took geography class). As for the third example, how we can tell is 1001 a cool number? What is “cool”? for whom? Base on the definition, the sentence “22 is an even number” is a statement, and this is a true statement. “44 is an odd number” is also a statement, but the false one. “1001 is a cool number!” is not a statement at all.

1. Which of the following sentences are statements?
 - a. When is the first day of school this year?
 - b. The 4th of July is Independence Day.
 - c. How beautiful is it!
 - d. Washington, DC is a capital of the United States.
 - e. The sum of five and three.
 - f. Three times five is twenty-six.

2. Which of the following statements are true, and which are false?
 - a. There are 31 days in each January.
 - b. There are 28 days in each February.
 - c. Sunday is followed by Tuesday.
 - d. There are 7 days in each week.
 - e. There are 7 letters in the word “table”
 - f. The sum of all single digit natural numbers is equal to 45.
 - g. Every 3-digit natural number is greater than 100.
 - h. There is a greatest 5-digit natural number.
 - i. There is a greatest natural number.
 - j. There is a smallest natural number.

Let's take a look at the statement "New York City is the capital of the United States". We, definitely, can say is it True or False. Of course it's not true, we all know that the capital of the US is Washington, DC. So we can say "it is not true, that New York City is the capital of the US", or, in a little more usual language, "New York City isn't the capital of the US". The last statement is a true statement.

"New York City is the capital of the United States" False
 "New York City isn't the capital of the US" (negation) True

If the statement is True, its negated version has to be False and vice versa. They can't be both True or both False. This rule of the math logic is one of the oldest and is called "The law of the excluded middle".

3. Let's try to construct negation of the several statements.

- a. Number 111111111 is a prime number.
- b. There is nothing on the table.
- c. 0.5 and $\frac{1}{2}$ are not equal.
- d. The area of a rectangle is equal to the product of its length and width.
- e. Sum $18 \cdot 946 + 456$ is divisible by 9.
- f. $45784 > 45784$
- g. $345 < 12345$
- h. All birds can fly.
- i. All marine animals are fish.
- j. Some students like math.
- k. All natural numbers are divisible by 3.
- l. Penguins live on the North Pole.
- m. Polar bears live on the South Pole.

4. Using the law of the excluded middle prove, that the negation of statement was made incorrectly.

	Statement	Negation
1	All cats are gray.	All cats are not gray
2	Some berries are sweet.	Some berries are not sweet.
3	There are 30 days in some months.	There are no 30 days in some months.
4	All birds can fly.	There are no birds that can fly.

Categorical statements are the statements about the relationship between categories or classes of objects. It states whether one category is fully contained with another, is partially contained (there are at least one member of the category) within another or is completely separate.

"Any (all) natural number is divisible by 3" is a categorical statement and it is a false statement.

It is very easy to show: 5 is a natural number and it is not divisible by 3. It means that not any natural number is divisible by 3, some of them are not divisible by 3 and only one such number is enough to prove that the statement is false.

"The sum of any even numbers is an even number" is a categorical statement. The statement is about the category "sum of two arbitrary even numbers", this category belongs to the set of even numbers. We can either prove it wrong by showing at least one example of the odd sum of two

even number, or prove it true by reasoning. It is not enough to show several examples to prove that this statement is true and of course there are no example to prove it wrong.

Prove. Any even number can be represented as $2k$ (or $2n$) where $k, n \in N$ because in this case the number is even.

$$2k + 2n = 2(k + n), \quad k, n \in N$$

So, the sum is divisible by 2, or even number.

How the negation of the categorical statement can be done?

If the statement is about the whole category (all elements of the category) which forms a subset (we can use the set theory formalism here) of another category, the negated statement will show the existence of at least one element of the set, which doesn't belongs to the category.

“All birds can fly.” – the statement is telling us that the whole category (all birds) is belongs to another category, things that can fly. The negation of this statement should tell us that there is at least one element (one kind of birds) which can't fly. We can formulate it as :

“Some birds can't fly” or as “There are birds that can't fly.”

$P =$ “All men are not bald” , $\neg P =$ “Some men are bald” and vice versa:

$P =$ “Some cats are gray”, $\neg P =$ “All cats are not gray”

We can create more complex statements, for example

I like math and physics. (I like math AND I like physics.)

In which case this statement will be a true statement?

1. I like math, but I don't like physics.
2. I like physics, but I don't like math.
3. I like both, math and physics.
4. I don't like math, and I don't like physics.

To negate the statement, we have to make the negation which is false. What do you think such statement will look like?

In this case there are two statements, first is “I like math” (statement A) and the second is “I like physics” (statement B). Let's create a Truth table for our statement **A AND B**

A	B	A \wedge B (AND, \cdot)
T	T	T
T	F	F
F	T	F
F	F	F

The negation of each of the statements A and B are “I don't like math”, “I don't like physics”.

Truth table of $\neg A$ (not A) OR $\neg B$ (not B). In this case the whole statement is true if at least one of the statements is true (or both statements are true).

$\neg A$	$\neg B$	$\neg A \vee \neg B$ (OR, +)
T	T	T
T	F	T
F	T	T
F	F	F

From this we can see that the negation of the statement “I like math and physics” should be true in the case when the negation of one of the two statements or both are true. We came up with another logical symbol.

Statement P = ”I eat pear or apple after diner” = $A+B$.

Statement A = ”I eat pear after diner”, statement B = ”I eat apple after diner”.

A	B	A OR B (\vee)
T	T	T
T	F	T
F	T	T
F	F	F

In real life you will probably say “I eat either pear or apple after diner”. The phrase “either...or” in language usually means one out of two possibilities, not both, but in math logic the statement “Either A or B” is true if A is true, B is true, both A and B are true”.

Exercises:

- Which of the following statements are categorical statements?
 - Some type of plants and animals are listed in the list of endangered species.
 - All planets of the Solar system are rotating around the Earth in the same direction.
 - Some butterflies are yellow.
 - There are 22 books on the shelf.
 - Any natural number is greater than 0.
- Find one counter example to prove the following statements wrong:
 - All natural numbers are greater than 1.
 - Any number divisible by 5 ends with digit 5.
 - All rivers of the United States flow into the Pacific Ocean.
 - All marine animals are fish.
 - All American cities lie south of 50° latitude line.
- Mother told Mary, that she can play videogames if she will do her homework, also will do her room, and will do dishes after diner. Will Mary play videogames if she
 - Did her room?
 - Did her room and dishes?
 - Did her homework?
 - Did her homework and dishes?
 - Did all three assignments?

4. On the other day mother told Mary that she will play videogames if she will do her homework or will do dishes. Will Mary play the videogames if she
 - a. Did her homework?
 - b. Did the dishes?
 - c. Did both assignment?
5. The following statements are proven. Can you tell which prove is right and which is wrong?
 - a. All natural numbers are divisible by 7, for example: $14:7=2$.
 - b. Some proper fractions have denominator equals to 8, for example: denominator of the fraction $\frac{3}{8}$ is 8.
 - c. There are even numbers multiple of 3, for example 36 is multiple of 3, $36:3=12$
 - d. Some nouns in English language contain 5 letters, for example “table”.
 - e. All verbs in English language start with letter “w”, for example “(to) write”.
 - f. There are books written by Joanne Rowling, for example *Harry Potter and the Philosopher's Stone* .
6. Inhabitants of the city A always say the true, inhabitants of the city B always lie, and inhabitant of the city C say true or lie every other time. Fire department got a call from somebody: “We got a fire here, come over as soon as you can”. “Where is the fire?”, - the fireman asked. “In the city B”, - was the answer. Where should they go, if the fire in one of the cities is real?

Geometry.

During the first part of the class we learned about statements, negation, how to construct complex statements with AND and OR logical operators. There is another complex logical statement which can be constructed from two simple statements A and B: “If A then B”. If A is a true statement, then truthfulness of B will follow. We will discuss logic more next week, but now let’s use it in geometrical theorems. Two weeks ago, we proved theorem “In isosceles triangle angles at the base are equal”. There two statements here:

If the triangle is an isosceles triangle, then its angles at the base are equal. (1)

What about the reverse position of these two statements?

If two angles of the triangle are equal, then this triangle is an isosceles triangle.

Does it necessarily follow from the theorem (1)? We already saw the example:

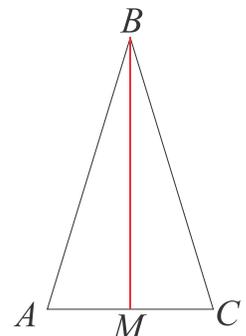
If two triangles are congruent, then all corresponding angles are equal, but the inverse predicate (this word also can be used for statement) is not true. From the statement “If corresponding angles of two triangles are equal”, doesn’t follow that two triangles are congruent.

Let’s prove that

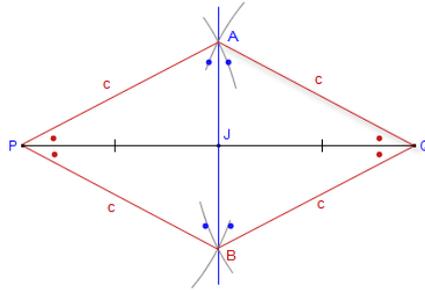
if two angles of a triangle are equal, triangle is an isosceles triangle.

In the triangle ABC $\angle BAC = \angle BCA$. Prove, that triangle ABC is an isosceles triangle. To prove this we need to prove that $|AB| = |BC|$ (or $[AB] \cong [BC]$). Let’s draw the segment BM, a bisector of the angle $\angle ABC$

$\triangle ABM \cong \triangle MBC$, because angle $\angle A = \angle C$, $\angle ABM = \angle MBC$ (BM is a bisector), therefore $\angle BMA = \angle BMC$ and the side BM is a common side. Two triangles are congruent by the ASA test. Therefore $|AB| = |BC|$.



How to divide the segment by half with the compass and a straight edge?



	Argument	Reason
1	Line segments AP, AQ, PB, QB are all congruent	The four distances were all drawn with the same compass width c.
2	Triangles $\triangle APQ$ and $\triangle BPQ$ are isosceles	Two sides are congruent (length c)
3	Angles AQJ, APJ are congruent	Base angles of isosceles triangles are congruent
4	Triangles $\triangle APQ$ and $\triangle BPQ$ are congruent	Three sides congruent (sss). PQ is common to both.
5	Angles APJ, BPJ, AQJ, BQJ are congruent. (The four angles at P and Q with red dots)	Corresponding parts of congruent triangles are congruent
6	$\triangle APB$ and $\triangle AQB$ are isosceles	Two sides are congruent (length c)
7	Angles QAJ, QBJ are congruent.	Base angles of isosceles triangles are congruent
8	Triangles $\triangle APB$ and $\triangle AQB$ are congruent	Three sides congruent (sss). AB is common to both.
9	Angles PAJ, PBJ, QAJ, QBJ are congruent. (The four angles at A and B with blue dots)	Corresponding parts of congruent triangles are congruent
10	Triangles $\triangle APJ$, $\triangle BPJ$, $\triangle AQJ$, $\triangle BQJ$ are congruent	Two angles and included side (ASA)
11	The four angles at J - $\angle AJP$, $\angle AJQ$, $\angle BJP$, $\angle BJQ$ are congruent	Corresponding parts of congruent triangles are congruent
12	Each of the four angles at J are 90° . Therefore AB is perpendicular to PQ	They are equal in measure and add to 360°
13	Line segments PJ and QJ are congruent. Therefore AB bisects PQ.	From (8), Corresponding parts of congruent triangles are congruent