

Geometry.

In **geometry**, two figures or objects are **congruent** if they have the same **shape** and size, or if one has the same shape and size as the **mirror image** of the other.^[1]

More formally, two sets of **points** are called **congruent** if, and only if, one can be transformed into the other by a combination of **rigid motions**, namely a **translation**, a **rotation**, and a **reflection**. This means that either object can be repositioned and reflected (but not resized) so as to coincide precisely with the other object. So, two distinct plane figures on a piece of paper are congruent if we can cut them out and then match them up completely. Turning the paper over is permitted.

In elementary geometry the word *congruent* is often used as follows. The word *equal* is often used in place of *congruent* for these objects.

- Two **line segments** are congruent if they have the same length.
- Two **angles** are congruent if they have the same measure.
- Two **circles** are congruent if they have the same diameter.

Sufficient evidence for congruence between two triangles can be shown through the following comparisons:

- **SAS** (Side-Angle-Side): If two pairs of sides of two triangles are equal in length, and the included angles are equal in measurement, then the triangles are congruent.
- **SSS** (Side-Side-Side): If three pairs of sides of two triangles are equal in length, then the triangles are congruent.
- **ASA** (Angle-Side-Angle): If two pairs of angles of two triangles are equal in measurement, and the included sides are equal in length, then the triangles are congruent.

SAS (Side-Angle-Side).

ABC and $A'B'C'$ are two triangles such that

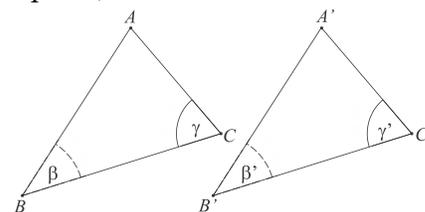
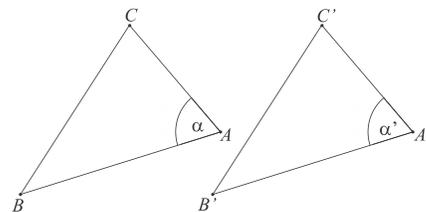
$AC = A'C'$, $AB = A'B'$, and $\angle A = \angle A'$ We need to prove that these triangles are congruent. Superimpose ΔABC onto $\Delta A'B'C'$ in such a way that vertex A would coincide with A' , the side AC would go along $A'C'$, and side AB would lie on the same side of $A'C'$ as $A'B'$.

Since AC is congruent to $A'C'$, the point C will merge with point C' . Due to the congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$ and due to the congruence of these sides, the point B will coincide with B' . Therefore the side BC will coincide with $B'C'$.

ASA (Angle-Side-Angle) ABC and $A'B'C'$ are two triangles, such that

$$\angle C = \angle C', \angle B = \angle B', \text{ and } BC = B'B'.$$

We need to prove, that these triangles are congruent. Superimpose the triangles in such a way that point C will coincide with point C' , the side CB would go along $C'B'$ and the vertex A would lie on the same side of $C'B'$ as A' . Then: since CB is congruent to $C'B'$, the point B will merge with B' , and due to congruence of the angle $\angle B$ and $\angle B'$, and $\angle C$ and $\angle C'$, the side BA will go along $B'A'$, and side CA will go along $C'A'$. Since two lines can intersect only at 1 point, the vertex A will have merge with A' . Thus, the triangles are identified and are congruent.



Theorem. In isosceles triangle the bisector passed to the base (in isosceles triangle the base is the side different from two equal sides) is a median and an altitude as well.

Let the triangle $\triangle ABC$ be an isosceles triangle, such that $AB = BC$, and BM is a bisector. We need to prove that BM is a median and an altitude, which means that $AM = MC$ and angle $\angle BMC$ is a right angle.

BM is a bisector, so $\angle ABM$ and $\angle MBC$, the triangle $\triangle ABM$ is an isosceles triangle, so $AB = BC$ and the segment MB is common side for triangles $\triangle ABM$ and $\triangle MBC$. Based on the Side-Angle-Side criteria, the triangles $\triangle ABM$ and $\triangle MBC$ are congruent. Therefore, $AM = MC$ (BM is a median), angles $\angle A$ and $\angle C$ are congruent. (Isosceles triangle has equal angles adjacent to the base).

$\angle A + \angle B + \angle C = 180^\circ = 2\angle A + \angle B \Rightarrow 90^\circ = \angle A + \frac{1}{2}\angle B$ but for the triangle ABM (as well as for MBC), $\angle A + \frac{1}{2}\angle B + \angle BMA = 180^\circ$, therefore $\angle BMA = 90^\circ$ and BM is also an altitude.

Exercise.

On one side of an angle $\angle A$, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ met on the bisector of the angle $\angle A$.

