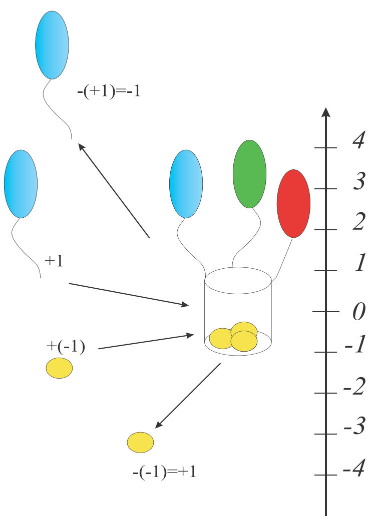
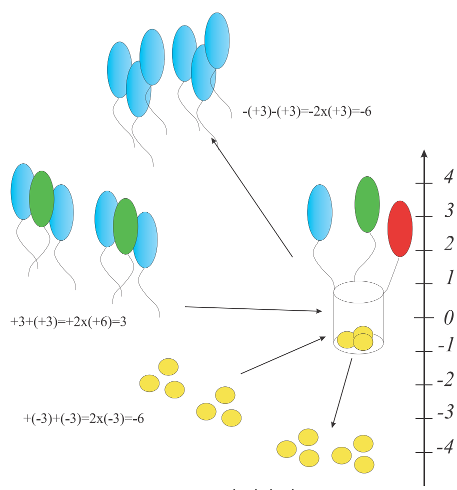
Accelerated Math 1. Class work 5.

Algebra.

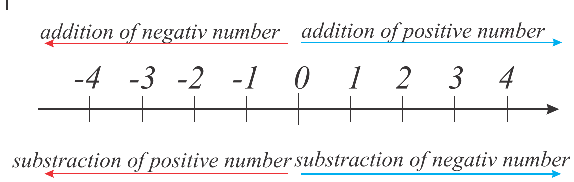
1. Positive and negative numbers.

Negative numbers represent opposites. If positive represents movement to the right, negative represents movement to the left. If positive represents above sea level, then negative represents below sea level. If positive represents a deposit, negative represents a withdrawal. They are often used to represent the magnitude of a loss or deficiency. Negative numbers appeared for the first time in history in the Nine Chapters on the Mathematical Art, which in its present form dates from the period of the Chinese Han Dynasty (202 BC – AD 220), but may well contain much older material. Liu Hui (c. 3rd century) established rules for adding and subtracting negative numbers. By the 7th century, Indian mathematicians such as Brahmagupta were describing the use of negative numbers. Islamic mathematicians further developed the rules of subtracting and multiplying negative numbers and solved problems with negative coefficients. Western mathematicians accepted the idea of negative numbers by the 17th century. Prior to the concept of negative numbers, mathematicians such as Diophantus considered negative solutions to problems "false" and equations requiring negative solutions were described as absurd.



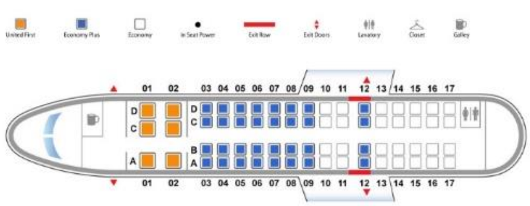


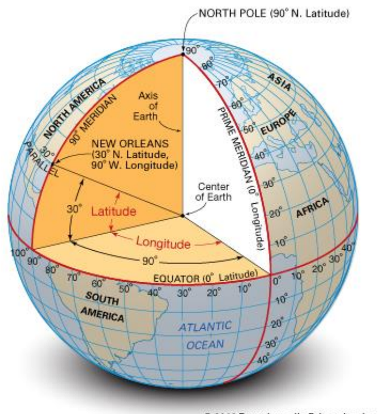
Two numbers that have the same magnitude but are opposite in signs are called opposite numbers.



2. Positive and negative numbers. Absolute value of a number.

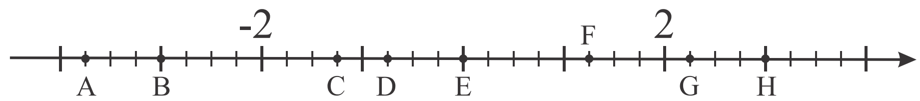
Coordinates are a set of values that show an exact position of an object. How many variables do we need to find our place in a theater? In a plane? What we can use as values?





How many different variables do we need to find our position on the surface of the Earth? How many values do we need to show the exact position of the point on a number line?

Find the coordinates of points A, B, C, D, E, F, G, and H on the number line below:



Mark the points A(0), B(1), C$\left(-1\frac{1}{2}\right)$, D$(5)$, E$(-5)$, F$(-3)$, G$(3)$

Is there anything in common between points F and G, points D and E?

On a straight line we can't say anything about the position of each point, we need to have a refence, point of origin, and we also need to know the length of a unit segment. After we choose the position of 0 and the length of the unit segment we can find a very important information about each point, the distance between the point and the origin, 0. By choosing the direction of increase of numbers, we define the sign (positive or negative) of all numbers. For each point on a number line we can assign a number, negative or positive, and each of this number can be described by the distance from 0 and the sign. This distance is called an absolute value of a number. And, of cause, it is always positive.

$$\begin{cases} |a| = a, & if\ a \geq 0 \\ |a| = -a, & if\ a < 0 \end{cases}$$

$|5| =$ $\quad$ $|-5| =$ $\quad$ $|10| =$ $\quad$ $|-10| =$

Does a fraction have an absolute value?

$\left|\frac{1}{2}\right| =$ $\quad$ $\left|-\frac{1}{2}\right| =$

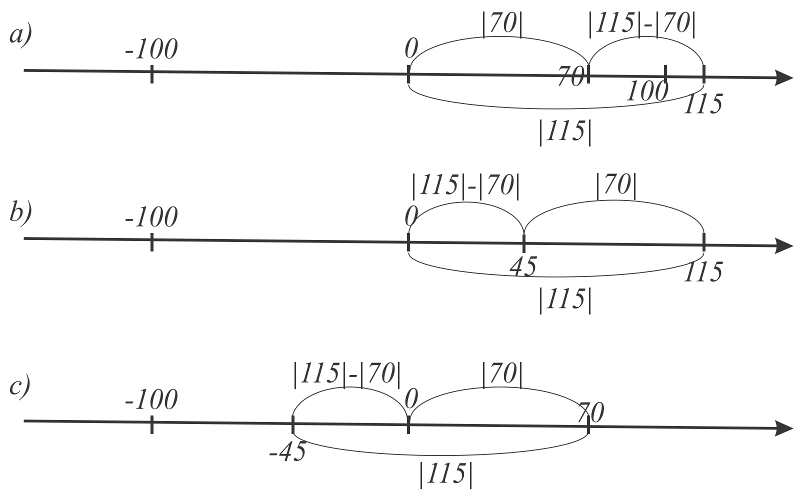
Can we solve the following equation? How many solutions does it have.

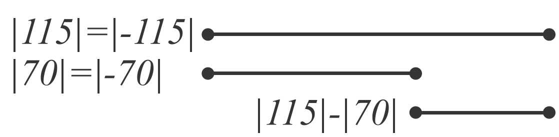
$$|x| = 5$$

To solve an equation means to find all possible values which will give us a true statement when put into the equation instead of a variable.

$|x| = 3$ $\quad$ $|y| = 10$ $\quad$ $|z| = -2$

a) Now let's try to add 2 numbers, one positive and one negative. (Subtraction is always can be seen as an addition of a negative number; subtraction of a negative number, as we all know, is an addition of a positive number.) As an example we will add 115 and (-75) and 75 and (-115) . In both cases absolute values of summands are the same, but in the first case the absolute value of the negative number is smaller than that of the positive summand and in the second case absolute value of the negative summand is greater. Look at the represents absolute values of $|115| = |-115|$, $|70| = |-70|$ and their difference $|115| - |70|$. Another visual representation of the same absolut valus is shown on the picture below.



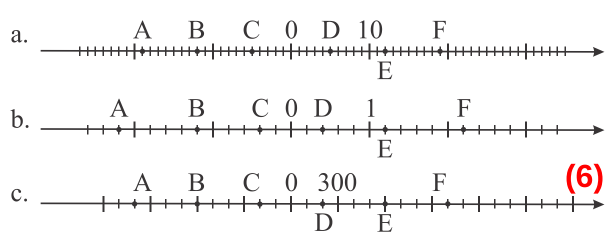


b) represents subtraction of 70 (or addition of -70) from (to) 115. If we subtract (or add negative) number which absolute value is smaller from the one which absolute value is grater we will do the usual operation of subtraction as we did before when we only operated with natural numbers.

c) represents the operation of subtraction (or addition of a negative number) of a number which absolute value is grater then the abs. value of the numbers from which we are subtracting. The **absolute value** of the result will be exactly the same, as in previous example, but the result itself will be negative, opposite to the result in the previous example.

Exercises.

1. Write the coordinates and absolute values of the coordinates of the points A, B, C, D, E, F marked on the number lines below:



(6)

2. Compare:

$|7+3| \quad |7|+|3|$ $\qquad$ $|7-3| \quad |3-7|$ $\qquad$ $|7-3| \quad |7|-|3|$

$|3-7| \quad |3|-|7|$ $\qquad$ $|a-b| \quad |b-a|$ $\qquad$ $|a+b| \quad |a|+|b|$

$|7-3| \quad |7|+|3|$ $\qquad$ $|3a| \quad 3\cdot|a|$ $\qquad$ $|b\cdot a| \quad b\cdot|a|$

(6)

3. Compute:
   a. $465-283=$
   b. $253-465=$
   c. $89-121=$
   d. $121-89=$

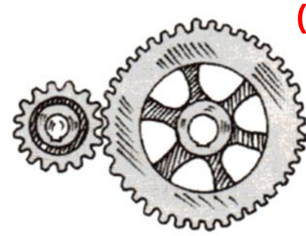
(6)

4. Open parenthesis, combine like terms and simplify the expression:

$$c-(c-d)-\left(c-\frac{d}{2}\right)-\left(c-\frac{d}{4}\right)-\left(c-\frac{d}{8}\right)-\left(c-\frac{d}{16}\right)+\frac{d}{16}$$

(6)

5. Two gears are in in clutch. One gear has 18 cogs, and another has 63. How many turns will each gear make before they both return to their original position? **(6)**



6. If $a>b$, then $|a|>|b|$. Is this statement always a true statement? **(4)**
7. If $a<b$, then $|a|>|b|$. Is this statement always a true statement? **(4)**
8. Evaluate:

Hint : how to add all numbers from 1 to 100: $1+2+3+4+\cdots+98+99+100$
Let's add these numbers twice:
$1+2+3+4+\cdots+98+99+100+1+2+3+4+\cdots+98+99+100$ using commutative property of addition we can rewrite this double sum this way: **(4)**

$(1+99)+(2+98)+(3+97)+(4+96)\ldots+(98+2)+(99+1)+100+100$
$\qquad = 99\cdot100+100+100=101\cdot100=10100$

There are 99 pairs of numbers sum of which is equal to 100, and 200, altogether this expression is 10100. But it's twice more then the initial sum of numbers from 1 to 100. The result is $10100:2=5050$.

Another way to look at this problem:

$1+2+3+4+\cdots+98+99+100=(1+99)+(2+98)+\cdots+(49+51)+50+100$
$\qquad = 49\cdot100+100+50=4900+100+50=5050$

   a. $1+(-2)+3+(-4)+\cdots+9+(-10)$;
   b. $1+(-2)+3+(-4)+\cdots+99+(-100)$;
   c. $(-1)+2+(-3)+4+\cdots+(-9)+10$;
   d. $(-1)+2+(-3)+4+\cdots+(-99)+100$;

9. To the number a add the number, opposite to b

$a=12, b=-7$;	$a=-24, b=-13$;	$a=-14, b=7$;
$a=24, b=13$;	$a=13, b=16$;	$a=-16, b=-18$

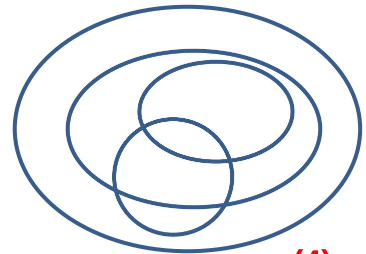
(4)

10. Positive or negative number will be the product of
    a) Two negative and one positive numbers.
    b) One negative and two positive numbers
    c) Three negative numbers.

(4)

Review

11. On the picture 4 Venn diagrams are drawn. They represent sets A, B, C, and D.



A is a set of proper fractions,
B is a set of proper fractions with denominator 8,
C is a set of fractions with nominator 5,
D is a set of all fractions.
Mark the sets A, B, C, and D. Put the following fractions into the right set:

$$\frac{2}{9}, \quad \frac{15}{7}, \quad \frac{3}{8}, \quad \frac{5}{8}, \quad \frac{5}{16}, \quad \frac{5}{4}$$

(4)

12. Prove that
    a. $43 \cdot 15 - 55 \cdot 15 + 34 \cdot 15$ is divisible by 22
    b. $12 \cdot 17 - 16 \cdot 17 + 13 \cdot 17$ is divisible by 9
    c. $99 \cdot 51 - 99 \cdot 91 + 69 \cdot 99$ is divisible by 29
    d. $63 \cdot 23 - 32 \cdot 63 + 22 \cdot 63$ is divisible by 13

(4)

13. Simplify the following expressions:
    a. $2 + 3a + xy + 4 - a + xy - 6;$
    b. $d - 4 + t + t + 32 + 3d;$
    c. $x + 5s - 3s + 2x;$
    d. $4x - \left(3x + (2x + 1)\right);$
    e. $y - \left(2y - (3y - 4)\right);$
    f. $z - \left(2z + \left(3z - (4z + 5)\right)\right);$

(6)

14. Simplify the following fractions:

$$\frac{12 \cdot 5 + 12 \cdot 9}{12 \cdot 21} =$$

$$\frac{14 \cdot 5 - 14 \cdot 2}{28} =$$

$$\frac{8 \cdot 8 - 8 \cdot 7}{8 \cdot 5} =$$

$$\frac{19 \cdot 8 - 19 \cdot 6}{38} =$$

(6)

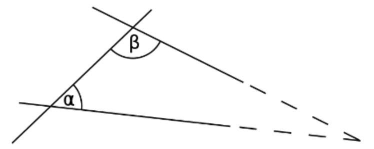
Geometry.

Two of the most important building blocks of geometric proofs are axioms and postulates. Axioms and postulates are essentially the same thing: mathematical truths that are accepted without proof. Their role is very similar to that of undefined terms: they lay a foundation for the study of more complicated geometry. One of the greatest Greeks (*Euclid*) achievements was setting up such rules for plane geometry. This system consisted of a collection of undefined terms like point and line, and five axioms from which all other properties could be deduced by a formal process of logic. Four of the axioms were so self-evident that it would be unthinkable to call any system a geometry unless it satisfied them:

1. A straight line may be drawn between any two points.
2. Any terminated straight line may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.

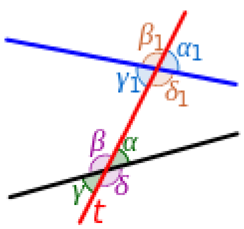
But the fifth axiom was a different sort of statement:

5. If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.

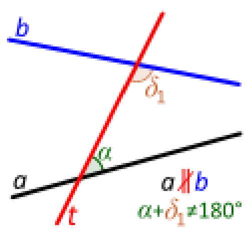


Mathematicians found alternate forms of the axiom that were easier to state, for example:

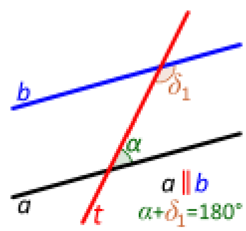
5'. For any given point, not on a given line, there is exactly one line through the point that does not meet the (or parallel to) given line.



Eight angles of a transversal. (Vertical angles such as γ and α are always congruent.)

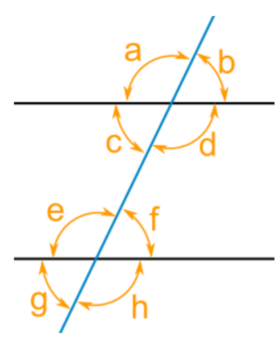


Transversal between non-parallel lines. Consecutive angles are not supplementary.



Transversal between parallel lines. Consecutive angles are supplementary.

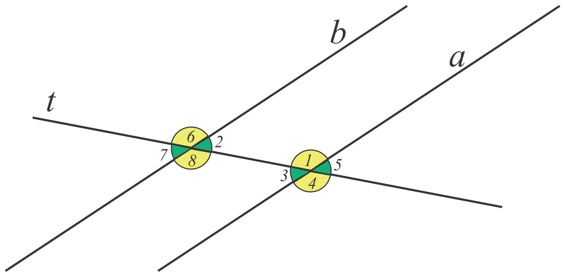
Transversal
Parallel Lines
Vertical Angles
Corresponding Angles
Alternate Interior Angles
Alternate Exterior Angles
Consecutive Interior Angles



The Euclid's *Elements* also include the following five "common notions":

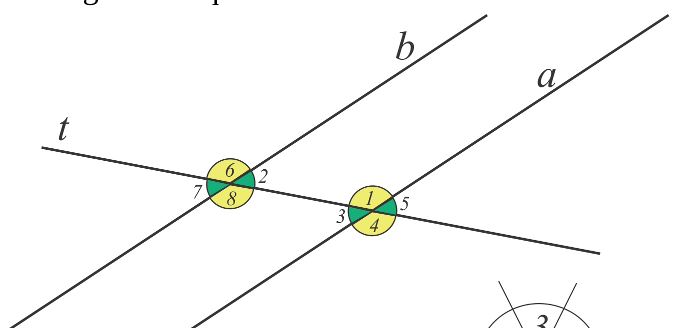
1. Things that are equal to the same thing are also equal to one another (formally the Euclidean property of equality, but may be considered a consequence of the transitivity property of equality).
2. If equals are added to equals, then the wholes are equal (Addition property of equality).
3. If equals are subtracted from equals, then the remainders are equal (Subtraction property of equality).
4. Things that coincide with one another are equal to one another (Reflexive Property).
5. The whole is greater than the part.

5^{th} postulate:



2 lines are crossed by transversal. If 2 consecutive interior angles formed by this transversal are supplementary (add up to a straight angle), then these 2 lines are parallel. (if they add up to an angle less than straight angle, these 2 lines will intersect on this side). Also, if two lines are parallel, 2 consecutive interior angles are supplementary. Base on this postulate, a few theorems can be proved:

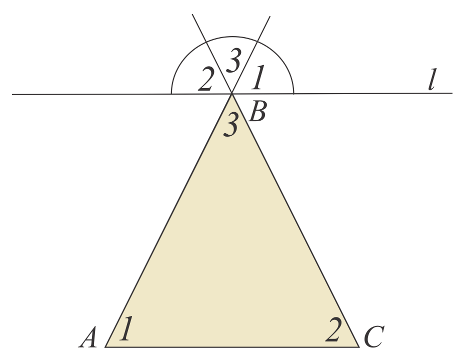
1. If 2 parallel lines crossed by transversal, consecutive angles are equal.
   $\angle 1=\angle 6$, $\angle 7=\angle 3$, and so on.
   Proof:
   $\angle 2+\angle 1=180$ (straight angle) by 5th postulate, $\angle 2+\angle 6=180$ (straight angle) $\Rightarrow$ $\angle 6=\angle 1$
   (Converse theorem are also can be proved:
   If consecutive angles formed by transversal are equal, two lines are parallel)
2. If 2 parallel lines crossed by transversal, alternate angles are equal.
   $\angle 2 = \angle 3$, $\angle 1 = \angle 8$.
   Proof:
   $\angle 2+\angle 1=180$ (straight angle) by 5th postulate, $\angle 1+\angle 3=180$ (straight angle) $\Rightarrow$ $\angle 2=\angle 3$



Also, the theorem of the angles of a triangle:

Three angles of any triangle sum to a straight angle.

Line *l* is parallel to line AC. Angles (3) are equal as vertical angles, angles (2) are equal and angles (1) are equal because line *l* is parallel to line AC.

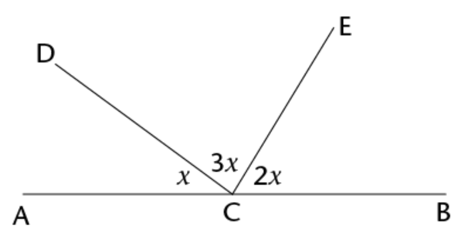


Proof:
Let's draw a line l, parallel to the one side of the triangle. Based on the previous theorems, proved above, we can see that three angles ad up to a straight angle.

Exercises.

15. On a line mark two points. How many segments were formed? Add one point. How many segments are there now? Add one more point. How many segments are there now? How many segments 6 points will form on the line? 10? 99? **(10)**
16. Calculate the measure of angle x from the picture below

    (points A, C and B lie on the same line)



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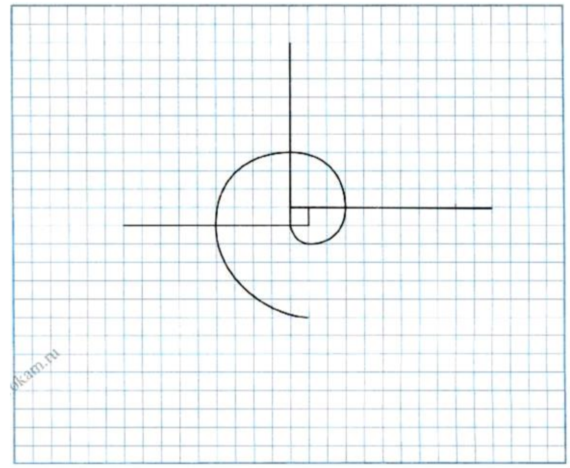
17. Continue the spiral curve. Hint- each element of the line is a quoter of a circle. Draw the picture in your notebook, use a compass and a ruler!



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