MATH 7

ASSIGNMENT 20: QUADRATIC EQUATION SUMMARY

Quadratic equations: summary

Recall from last time

- A quadratic polynomial is expression of the form $p(x) = ax^2 + bx + c$.
- Roots of quadratic polynomial are numbers such that $p(x) = 0$. If x_1, x_2 are roots, then $p(x) = a(x - x_1)(x - x_2)$.
- Vieta formulas: if x_1, x_2 are roots of $x^2 + bx + c$, then

$$x_1 + x_2 = -b$$
$$x_1 x_2 = c$$

- Completing the square: we can rewrite

$$ax^2 + bx + c = a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{D}{4a} = a\left(\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{D}{4a^2}\right) \tag{1}$$

where $D = b^2 - 4ac$.

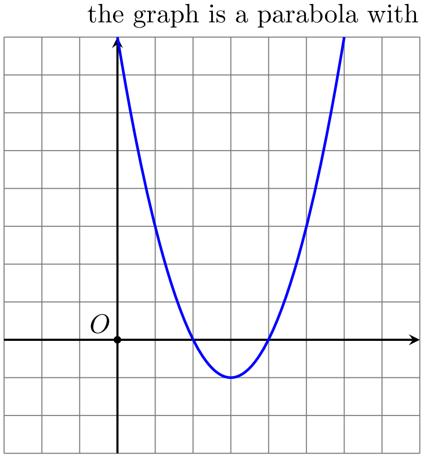
From this, one gets the quadratic formula: if $D < 0$, there are no roots; if $D \geq 0$, then the roots are

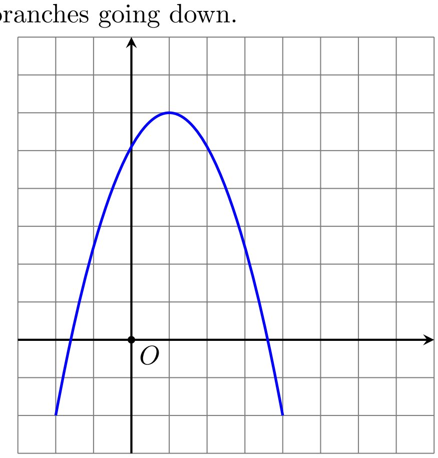
$$x_1, x_2 = \frac{-b \pm \sqrt{D}}{2a}$$

Graphs

From formula (1), we see that:

- If $a > 0$, then the smallest possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going up.
- If $a < 0$, then the *largest* possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going down.





Inequalities

To solve inequality of the form $ax^2 + bx + c > 0$: first, find the roots. Then,

- if $a > 0$, $p(x) > 0$ for $x < x_1$ and for $x > x_2$, and $p(x) < 0$ between the roots
- if $a < 0$, $p(x) < 0$ for $x < x_1$ and for $x > x_2$, and $p(x) > 0$ between the roots

Homework

1. For what values of a does the polynomial $x^2 + ax + 14$ has no roots? exactly one root? two roots?

2. Solve the following equations and inequalities. For each polynomial, also sketch the graph.

(a) $x^2 - 5x + 4 < 0$ (b) $2x^2 + 5x - 3 > 0$ (c) $x^2 > 1 + x$

(d) $-x^2 + 2x - 4 > 0$ (e) $x^2 - x + 6 \geq 0$

3. Let x_1, x_2 be roots of equation $x^2 + 3x + 4 = 0$. Find
(a) $x_1^2 + x_2^2$ (b) $\frac{1}{x_1^2} + \frac{1}{x_2^2}$

4. Of all the rectangles with perimeter 4, which one has the largest area? [Hint: if sides of the rectangle are a and b, then the area is $A = ab$, and the perimeter is is $2a + 2b = 4$. Thus, $b = 2 - a$, so one can write A using only a....]

5. Prove that for any point P on the parabola $y = \frac{x^2}{4} + 1$, the distance from P to the x-axis is equal to the distance from P to the point $(0, 2)$.

6. This question is about making sense of the following (infinite) expression

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

(a) Compute the first several approximations (write the fractions in simplest form):

$$\frac{1}{1+1}; \qquad \cfrac{1}{1 + \cfrac{1}{1+1}}; \qquad \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}}$$

Can you guess the pattern?

*(b) Assuming that as we add more and more terms, we are getting closer to some number x, can we find x? [Hint: since adding one more step should not change x, we should have $x = \frac{1}{1+x}$...]

***7.** Find all intersection points of parabola $y = x^2$ and the circle with radius $\sqrt{6}$ and center at $(0, 4)$.