MATH 7: ASSIGNMENT 19
QUADRATIC EQUATION AND INEQUALITIES, VIETA FORMULA, ETC. GRAPHS.

VIETA FORMULAS

If a polynomial $p(x)$ of degree n has root x_1 (i.e., if $p(x_1) = 0$), then $p(x)$ is divisible by $(x - x_1)$, i.e. $p(x) = (x - x_1)q(x)$ for some polynomial $q(x)$ of degree $n - 1$. In particular, if x_1, x_2 are roots of quadratic polynomial $ax^2 + bx + c$, then $ax^2 + bx + c = a(x - x_1)(x - x_2)$. Expanding the right hand side of this equation we have Vieta formulas

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned}$$

Similarly, if polynomial of degree n $p(x) = ax^n + bx^{n-1} + \cdots + c$ has n roots $x_1, x_2, \ldots, x_n$ one can write $p(x) = a(x - x_1) \ldots (x - x_n)$. Expanding the right hand side we obtain Vieta formulas

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= -\frac{b}{a} \\ &\cdots \\ x_1 x_2 \ldots x_n &= \frac{c}{a} \end{aligned}$$

SOLVING POLYNOMIAL INEQUALITIES

We discussed the general rule for solving polynomial inequalities:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x - 1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- Roots $x_1, x_2, \ldots$ divide the real line into intervals; define the sign of each factor and the product on each of the sign intervals.
- If the inequality has $\geq$ or $\leq$ signs you should also include the roots themselves into the intervals.

Example 1: $x^2 + x - 2 > 0$. We find roots of the equation $x^2 + x - 2 = 0$ and obtain $x = -2, 1$. The inequality becomes $(x + 2)(x - 1) > 0$ and roots -2, 1 divide the real line into three intervals $(-\infty, -2), (-2, 1), (1, +\infty)$. It is easy to see that the polynomial $x^2 + x - 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then $x < -2$ or $x > 1$. We sometimes, write this also as $x \in (-\infty, -2) \cup (1, +\infty)$. (sign $\cup$ means "or").

Example 2: $-x^2 - x + 2 \geq 0$. We have $-(x + 2)(x - 1) \geq 0$. The left hand side is positive for $-2 < x < 1$. As the sign in the inequality is $\geq$ we have to includes the roots into the interval and obtain $-2 \leq x \leq 1$. One can also write $x \in [-2, 1]$ (square brackets here mean that the endpoints of the interval are included).

Example 3: $x^2 + x + 2 \geq 0$. The polynomial here does not have roots (the discriminant $1 2 - 4 \cdot 1 \cdot 2 < 0$). Therefore, the real line is not divided into the intervals, which means that the polynomial is of the same sign for all x. We check that it is positive, for example, for $x = 0$. The solution is that x is any number. We can write $x \in (-\infty, +\infty)$.

Example 4: $x^2 + x + 2 < 0$. The polynomial does not have roots and is positive everywhere. This means that the inequality does not have solutions at all. One can also write $x \in \emptyset$.

Example 5: $x^2 - 2x + 1 > 0$. The inequality is $(x - 1)^2 > 0$. There is only one root here which divides the real line into two intervals. The solution is $x < 1$ or $x > 1$, that is any x except for $x = 1$. One can write $x \in (-\infty, 1) \cup (1, +\infty)$.

Graphs of quadratic functions

1. We know how to draw the graph of $y = x^2$. It's a parabola.
2. We know that the graph of $y = x^2 + b$ can be obtained from the graph of $y = x^2$ by shifting up by b units (or down, if $b < 0$)
3. We know that the graph of $y = (x+a)^2$ can be obtained from the graph of $y = x^2$ by shifting *left* by a units (or right, if $a < 0$).
4. Based on the two fact above, we can draw a graph of any function of the type $y = (x+a)^2+b$.

We can transform any quadratic function $y = x^2 + px + q$ to $y = (x + a)^2 + b$ by completing the square.

Homework

1. Find the roots of the equation $4x^2 - 2x - 1 = 0$ **WITHOUT** using the formula for roots of quadratic equation. That is, complete the square and use the difference of squares formula to factorize the polynomial.

2. What is the sum of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$? What is the product of those roots? Can you guess the roots?

3. Without solving the equation $3x^2 - 5x + 1 = 0$ find the arithmetic mean of its roots (that is $\frac{x_1+x_2}{2}$) and their geometric mean (that is $\sqrt{x_1 x_2}$).

4. Solve the equation $x^4 - x^2 - 2 = 0$.

5. Solve the following equations and inequalities:
(a) $x^2 + 2x - 3 = 0$, $x^2 + 2x - 3 > 0$, $x^2 + 2x - 3 \le 0$
(b) $x^2 + 2x + 3 = 0$, $x^2 + 2x + 3 \ge 0$, $x^2 + 2x + 3 < 0$
(c) $-x^2 + 6x - 9 = 0$, $-x^2 + 6x - 9 \ge 0$, $-x^2 + 6x - 9 < 0$
(d) $3x^2 + x - 1 = 0$, $3x^2 + x - 1 \ge 0$, $3x^2 + x - 1 \le 0$

6. For a given a, what is the minimum of the value of the expression $x^2 - ax + 1$? At what x that expression has a minimum value?

7. Use completing the square method to draw the following graphs:
(a) $y = x^2 - 5x + 5$ (b) $y = x^2 - 4x + 2$ (c) $y = x^2 - x - 1$
(d) $y = -x^2 + 3x - 0.5$ (e) $y = x^2 + 4x - 45$