

MATH 7: ASSIGNMENT 16
QUADRATIC EQUATION

QUADRATIC EQUATIONS

Today we discussed how one solves quadratic equation:

$$ax^2 + bx + c = 0$$

The method used is called “completing the square”. Here is an example how it works:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 7 = (x + 3)^2 - 7 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

thus, $x^2 + 6x + 2 = 0$ if and only if $x + 3 + \sqrt{7} = 0$, which gives $x = -3 - \sqrt{7}$, or $x + 3 - \sqrt{7} = 0$, which gives $x = -3 + \sqrt{7}$.

The same trick works in general: if $a = 1$, then

$$(1) \quad \begin{aligned} x^2 + bx + c &= x^2 + 2\frac{b}{2}x + c = \left(x^2 + 2\frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c \\ &= \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{4} \end{aligned}$$

where $D = b^2 - 4c$.

Thus, $x^2 + bx + c = 0$ is equivalent to

$$\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$$

If a is not equal to 1, the answer is similar: $ax^2 + bx + c = 0$ is equivalent to

$$\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}, \quad D = b^2 - 4ac$$

Therefore, if $D < 0$, there are no solutions; if $D \geq 0$, solutions are

$$(2) \quad \begin{aligned} x + \frac{b}{2a} &= \pm \frac{\sqrt{D}}{2a} \\ x &= \frac{-b \pm \sqrt{D}}{2a} \end{aligned}$$

HOMEWORK

- (a) Use formula (1) to prove that for any x , $x^2 + bx + c \geq -D/4$, with equality only if $x = -b/2$.
(b) Find the minimal possible value of the expression $x^2 + 4x + 2$
(c) Given a number $a > 0$, find the maximal possible value of $x(a - x)$ (the answer will depend on a).

- Solve the following equations. Carefully write all the steps in your argument.

$$(a) \ x^2 - 5x + 5 = 0 \quad (b) \ \frac{x}{x-2} = x - 1 \quad (c) \ x^2 = 1 + x$$

$$(d) \ 2x(3 - x) = 1 \quad (e) \ x^3 + 4x^2 - 45x = 0$$

- If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$; $x^3 + \frac{1}{x^3}$.

- *4. Consider the sequence $x_1 = 1$, $x_2 = \frac{x_1}{2} + \frac{1}{x_1}$, $x_3 = \frac{x_2}{2} + \frac{1}{x_2}$... Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you guess this value? [Hint: solve equation $x = \frac{x}{2} + \frac{1}{x}$.]