MATH 7
ASSIGNMENT 13: COMBINATIONS

Formula for binomial coefficients

Recall numbers from pascal triangle $\binom{n}{k}$. These numbers appear in many problems:

$$\begin{aligned}{}_nC_k = &\text{ The number of paths on the chessboard going } k \text{ units up and } n-k \text{ to the right}\\ = &\text{ The number of words that can be written using } k \text{ zeros and } n-k \text{ ones}\\ = &\textbf{ The number of ways to choose } k \textbf{ items out of } n \textbf{ (order doesn't matter)}\end{aligned}$$

It turns out that there is an explicit formula for ${}_nC_k$:

$${}_nC_k = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Compare it with the number of ways of choosing k items out of n when the order matters:

$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

For example, there are $5 \cdot 4 = 20$ ways to choose to items out of 5 if the order matters, and $\frac{5\cdot 4}{2} = 10$ if the order doesn't matter.

Main formulas of combinatorics

- The number of ways to order k items is
$$k! = k(k-1)\cdots 2\cdot 1$$
- The number of ways to choose k items out of n if the order matters is
$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$
- The number of ways to choose k items out of n if the order does not matter is
$${}_nC_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!}$$

These numbers are the ones that appear in Pascal triangle and in many other problems:

$$\begin{aligned}{}_nC_k = &\text{ The number of paths on the chessboard going } k \text{ units up and } n-k \text{ to the right}\\ = &\text{ The number of words that can be written using } k \text{ zeros and } n-k \text{ ones}\end{aligned}$$

Problems

1. A senior class in a high school, consisting of 120 students, wants to choose a class president, vice-president, and 3 steering committee members. How many ways are there for them to do this?

2. Remember that a poker hand is a selection of 5 cards out of a 52-card deck (4 suits, 13 card ranks in each suit).

   How many poker hands are there that contain

   (a) Exactly two aces
   (b) Exactly 3 kings
   (c) Two aces and three kings
   (d) Exactly three cards of the same rank
   (e) At least one pair of the same rank [Hint: how many hands are there that contain no pairs?]
   (f) Three cards of one rank and 2 cards of another rank (in poker, this is called *full house*).

3. Remember that in one of the lotteries run by New York State, "Sweet Million", they randomly choose 6 numbers out of numbers 1–40. This week's winning numbers are 04-05-16-18-23-31

   If you had chosen 6 numberes at random, what are your chances that you have guessed correctly exactly 3 of them?

4. Nikita has 7 pieces of candy, and Lev has 9 (all different). They want to trade 5 pieces of candy. How many possibilities are there?

5. In how many ways can you cut a necklace consisting of 30 different beads into 8 pieces?

6. If you have 5 lines on the plane so that no two are parallel and there are no triple intersection points, how many triangles do they form? What if there are n lines?

7. Two persons, A and B, play the following game. They toss a coin 5 times. If they get exactly 2 or 3 heads, A wins 1 tugrik. Otherwise B wins 1 tugrik. Would you rather play for A or B?

8. A rook is placed on the leftmost square of a 1×30 strip of square ruled paper. At every turn, it can move any number of squares to the right.

   (a) How many ways are there for the rook to reach the rightmost square in exactly 5 turns?
   (b) How many ways are there for the rook to reach the rightmost square if there are no restrictions on number of turns?