MATH 7
ASSIGNMENT 6: VECTORS: CENTER OF GRAVITY

Application: center of gravity

For a collection of points $A_1, \dots, A_n$ and positive numbers $m_1, \dots, m_n$ (masses placed at these points), we define the center of gravity of this collection of points to be a point M such that

$$\overrightarrow{OM} = \frac{m_1 \overrightarrow{OA_1} + \dots + m_n \overrightarrow{OA_n}}{m_1 + m_2 + \dots + m_n}$$

It can be shown that this definition does not depend on the choice of point O: if we choose another point O' and define M' so that $\overrightarrow{O'M'} = \frac{m_1 \overrightarrow{O'A_1} + \dots + m_n \overrightarrow{O'A_n}}{m_1 + m_2 + \dots + m_n}$ then in fact $M = M'$.

Examples:

- Center of gravity of two points A, B with equal mass at them is the midpoint of the interval AB.
- Center of gravity of the four vertices of the parallelogram is the intersection point of its diagonals.

Problems

1. (a) Let masses $m_1 = 3$, $m_2 = 1$ be placed at points $A_1 = (3, 6)$, $A_2 = (11, 2)$. Find the center of gravity of these two masses. Does it lie on the segment A_1A_2? in what proportion does it divide it?

(b) Consider the center of gravity M of a system of two masses m_1, m_2 at points A_1, A_2. Prove that then $\overrightarrow{A_1M} = \frac{m_2}{m_1 + m_2} \overrightarrow{A_1A_2}$. Can you write a similar formula for $\overrightarrow{MA_2}$?

(c) Prove that the center of gravity M of a system of two masses m_1, m_2 at points A_1, A_2 lies on the segment A_1A_2 and divides it in proportion $m_2 : m_1$.

2. (a) Let M the center of gravity of three points A, B, C with unit mass at each of them. Prove that then

$$\overrightarrow{OM} = \tfrac{1}{3} \overrightarrow{OA} + \tfrac{2}{3} \overrightarrow{OA_1}$$

where A_1 is the midpoint of BC.

(b) Prove that all three medians of a triangle intersect at a single point M which divides each of them in proportion $2 : 1$

***3.** Consider a triangle $\triangle ABC$ and let

(a) A_1 be the point on side BC which divides it in proportion $2 : 3$,

(b) B_1 be the point on side CA which divides it in proportion $3 : 4$,

(c) C_1 be the point on side AB which divides it in proportion $2 : 1$

Prove that the lines AA_1, BB_1, CC_1 all intersect at a single point. [Hint: this point would be the center of gravity of appropriately chosen 3 masses at points A, B, C.]