CHAPTER 13

INVESTIGATING SEQUENCES AND SERIES

A sequence is a list of numbers, while a series is a sum of numbers.
Investigative tools for sequences: Compare each term to neighboring terms. Find any symmetry, repetition hidden in the sequence itself through which we can recreate a large part of the initial sequence.

13.1 Arithmetic sequence

An arithmetic sequence is a sequence of numbers such that the difference between any two consecutive terms is the same. This fixed difference is known as the common difference of the sequence.

Consider n terms of an arithmetic sequence with the first term $a_1 = a$, last term a_n, and common difference d. If we use our first investigative tool to obtain a general formula for the n-th term, i.e. $a_n = a + (n-1)d$ and $a_{n-1} = a_n - d$.

Then we can view its associated series as:

$$a + (a + d) + (a + 2d) + \cdots + (a + (n-1)d).$$

The general term is :

$$a_k = a + (k-1)d$$

$$a,\ a+d,\ a+2d\ \ldots,\ a_{n-1} = a_0 + (n-2)d,\ a_n = a_0 + (n-1)d,$$

To find the sum/series we should also look at the symmetry between the sequence and the sequence produced by reversing the order of the terms:

$$\begin{array}{c} a,\quad a+d,\quad a+2d,\ \ldots,\ a_n-2d,\ a_n-d,\ a_n, \\ a_n,\ a_n-d,\ a_n-2d,\ \ldots,\ \ a+2d,\quad a+d,\quad a. \end{array}$$

Investigate:

- Add the first term of the first sequence to the first term of the reversed sequence. Do the same with the second term of each of the sequences. What did you notice?

- Show that the average of all the terms in an arithmetic sequence is equal to the average of the first and last terms.

- Find a formula for the series (i.e. sum of the terms) in the original sequence.

The average of the first and last terms of an arithmetic sequence with n terms is

$$\frac{a + (a + (n-1)d)}{2} = \frac{2a + (n-1)d}{2}.$$

Thus, the sum of an n-term arithmetic series with first term a and common difference d is

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n(2a + (n-1)d)}{2}.$$

Homework Arithmetic Sequences

1. Graph the arithmetic sequences:(a) 1, 4, 7, 10, 13, 16 and (b)1,-1,-3,-5 . What type of graphs are they?
2. The first term of an arithmetic sequence is 5 and the 5th term is 33. Find the common difference of the sequence and find the 200-th term of the sequence.
3. In an arithmetic sequence $a_{10} = a_1 + 27$ and $a_2 = 7$. Find the sum of first 6 members of this sequence.
4. SchoolNova has twenty classrooms on a long corridor. If the first class is 10 feet away from the entrance and they are spaced 35 feet apart. In the morning, in the first class there is only one student, in the second class there are two other ones and in the 20th class there are twenty. What is total number of feet that kids have to make in order to arrive to their classrooms in the morning?
5. If the sum of the first $2n$ positive integers is 150 more than the sum of the first n positive integers, then find the sum of the first $3n$ positive integers.
6. Find the sum of

$$\sum_{k=-3}^{16} (2k + 10) = 4 + 6 + 8 + 10 + \cdots + 42.$$

7. Find a positive integer solution to the equation

$$\frac{1 + 3 + 5 + \cdots + (2n-1)}{2 + 4 + 6 + \cdots + 2n} = \frac{201}{202}.$$

13.2 Geometric sequences

A geometric sequence is a sequence of numbers for which there exists a constant r such that each term is r times the previous term. This constant r is called the common ratio of the sequence.

Consider n terms of an geometric sequence with the first term $a_1 = a$, last term a_n, and the common ratio of the sequence r. If we use our first investigative tool to obtain a general formula for the n-th term, i.e. the n^{th} term of the sequence is equal to ar^{n-1}.

Then we can view its associated series as:

$$T_n = a + (ar) + (ar^2) + \cdots + (a + r^{(n-1)}).$$

Investigate:

- By what can we multiply the terms of T_n to obtain another sequence that has many terms in common with T_n?
- Try to use the sum of terms of the newly obtained sequence and compute T_n in terms of $a, r,$ and n.

By multiplying the sequence $a, ar, ar^2, \cdots, ar^{n-1}$ by r, we create another sequence $ar, ar^2, \cdots, ar^{n-1}, ar^n$ that has many terms in common with S. We can compare their associated sums.

$$\begin{array}{rcccccccccccc} rT_n & = & & & ar & + & ar^2 & + & \cdots & + & ar^{n-1} & + & ar^n, \\ T_n & = & a & + & ar & + & ar^2 & + & \cdots & + & ar^{n-1}. & & \end{array}$$

Subtracting them gives

$$rT_n - T_n = ar^n - a$$

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The sum of a geometric series with n terms, first term a, and common ratio is

$$T_n = \frac{a(r^n - 1)}{r - 1}.$$

Investigate the sums of large sequences http://demonstrations.wolfram.com/PlotOfAGeometricSequenceAndItsPartialSums/

Homework Geometric Sequences

1. Graph the geometric sequences and their sums:
   (a) 1, 1/2, 1/4, 1/8, 1/16.
   (b) 2, 6, 18, 54.
   (c) 1, -1/2, 1/4, -1/8, 1/16.
2. The first term of a geometric sequence of positive numbers is 12, and the 7th term is 48. Find the 10^{th} term of the geometric sequence.
3. Compute $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^9}$
4. (general sequences) For a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = \dfrac{a_{n+1} + 1}{a_n}$ for $n \geq 1$ justify why the sequence is periodic and compute a_{2018} and a_{2018^2}

Review

1. How many ways are there to choose a gold medal winner, a silver medal winner, and a bronze medal winner in a group of 7 people?
2. In a right triangle, one leg is three times as long as the other, and the hypothenuse is equal to 10cm. What is the length of the larger leg?
3. Is the following statement true? If yes, prove by giving a truth table; if no, provide a counterexample and explain why it is not true: $NOT(A$ OR $B)$ is the same as (NOT A) OR (NOT B)
4. You draw three cards from the standard deck of 52 cards.
   (a) What are the chances that you will get no kings?
   (b) What are the chances that you will get exactly one king?
5. Draw the graphs of the functions: $y = 2x + 1$ and $y = |x - 1|$. Solve graphically the equation $2x + 1 = |x - 1|$
6. If $a + \frac{1}{a} = 11$ find $a^2 + \frac{1}{a^2}$
7. If $a + b = 1$ and $a^2 + b^2 = 9$, find $a^3 + b^3$.
8. Put n people 1,2,...,n on a Ferris wheel, one per seat. How many distinct ways of arranging them are there?