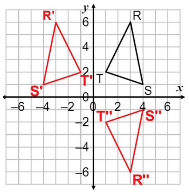
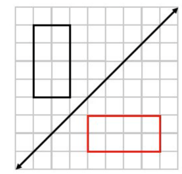


(a) Reflection about axis y.



(b) Reflection about the 1st diagonal y=x.

Figure 10.1: Reflections over the axis and diagonals

CHAPTER 10

MORE ISOMETRIES: REFLECTIONS, COMPOSITIONS

10.1 Reflections

Definition *A reflection about a line m is a transformation sending each point $P \in m$ onto itself and each point $P \notin m$ onto a point P' such that m is the perpendicular bisector of the segment PP'.*

For pairs of points P and Q both outside m we can build a triangle using a parallel-to-m line passing e.g. through P and a perpendicular-to-m line passing through Q. Let R be the intersection of these lines. Then we can show that $\Delta PQR \cong \Delta P'Q'R'$ where R' is the image by reflection of R. Let S be the middle of PP' (thus on m by definition of the reflection), and T the middle of QQ' (again on m). Then $STRP$ and $STR'P'$ are rectangles (because $PR//m//P'R'$ by our construction, and angles are right, by the reflection definition, etc). Thus $PS = RT = P'S = R'T$ and therefore $RQ = RT - QT = R'T - Q'T = R'Q'$. Using $PR = ST = P'R'$ (from rectangles), we conclude the triangle congruency (since they are right triangles), implying $PQ = P'Q'$, thus distance preservation.

If the line is an axis, or the diagonals we easily obtain the formulas for the reflected points.

- *When you reflect a point across the x-axis, the x-coordinate remains the same, but the y-coordinate changes signs. The reflection of the point (x,y) across the x-axis is the point (x,-y).*
- *When you reflect a point across the y-axis, the y-coordinate remains the same, but the x-coordinate changes signs. The reflection of the point (x,y) across the y-axis is the point (-x,y).*
- *When you reflect a point across the line $y = x$, the x-coordinate and y-coordinate change places. The reflection of the point (x,y) across the line $y = x$ is the point (y, x).*
- *If you reflect over the line y =-x, the x-coordinate and y-coordinate change places and are changing signs. The reflection of the point (x,y) across the line $y = -x$ is the point $(-y, -x)$*

Theorem *A reflection about a line m is an isometry: for pairs of points on m it clearly preserves their distance, and so it does for pairs of one point on m and one outside, by definition.*

The composition of a reflection with itself clearly leaves all points unchanged: $R_m(R_m(P)) = P$.
In the general case things can still be obtained using coordinate geometry. Let a point $P(x_0, y_0)$ and a line $y = mx + n$ Denote by P' the point mirrored over m.

Construct the perpendicular through P to y. It passes through P and L, the point of intersection between the line y and the perpendicular from P onto the line y. Double the length of the perpendicular in the direction of The endpoint is P' . First you have to get the perpendicular $y_p = m_p x + n_p$ (the dashed red line). We use that two perpendicular lines have negative reciprocal slopes $mm_p = -1$ As we know that P is on the line s, we simply obtain n_p by using that P verifies the equation of the line y_p, i.e. $y_0 = m_p x_0 + n_p$ To construct L we solve $y(x) = y_p(x)$ as the point belongs to both lines. We obtain the coordinates of the vector $PL = (x_L - x_P, Y_L - y_P$ and we obtain the coordinates of $P' = (x_L, y_L) + \vec{PL} = (x_L, y_L) + (x_L - x_P, Y_L - y_P)$

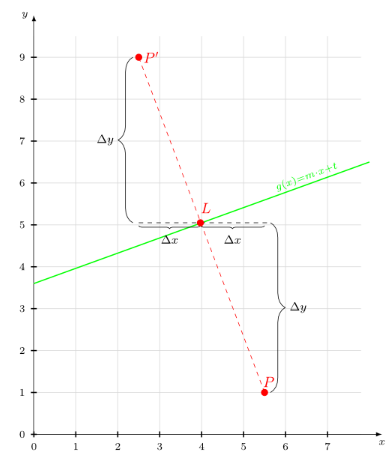
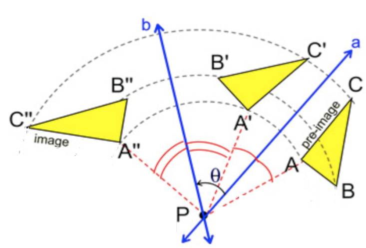


Figure 10.2: Reflections about a line

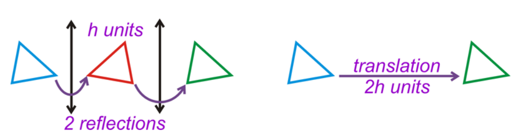
Theorem *The composition of two reflections about two intersecting lines a and b is a rotation $R_{O,2\widehat{(a,b)}}$.*

Let $P = a \cap b$. The vertices of the $\triangle ABC$ are not on a neither on b. Let $A'B'C'$ the image through the first reflection (about a) and $A''B''C''$ the image through the second reflection (about b). Then $\Delta PAA'$ and $\Delta PA'A''$ are isosceles, $\widehat{APA''} = 2\widehat{(a,b)}$ and $PA = PA' = PA''$, because a is the perpendicular bisector of AA' and b the perpendicular bisector of $A'A''$ (by definition of the reflection).

Theorem *The composition of two reflections about two different parallel lines is a translation of double the distance between the lines, along a direction perpendicular on them.*



(a) Reflection about intersecting lines



(b) Reflection about parallel lines

The pre-image and the two successive images are all colinnear because they belong to segments perpendicular on two parallel lines. Then we can simply add them up, and notice that the distance between the two lines is the sum of one half of each of these segments.

Theorem *The composition of a reflection R_m about a line m and a translation $T_{\vec{v}}$ with $\vec{v} \perp m$ is another reflection, about a different line q, such that $q//m$ and $d(q,m) = \frac{|\vec{v}|}{2}$.*

Let us apply Property 4 to these two reflections: $R_m(R_q(P)) = T_{\vec{v}}(P)$, because the construction of q satisfies its hypothesis. Let us then compose this equation once with R_q and then apply Property 2: $R_q(R_q(P)) = P$. We thus obtain $R_m(R_q(R_q(P))) = T_{\vec{v}}(R_q(P))$ which gives the result: $R_m(P) = T_{\vec{v}}(R_q(P))$

Theorem *Theorem: The composition of a reflection R_m and a rotation R_O with $O \in m$ is another reflection R_q with $q \cap m = \{O\}$.*

Let P not on m, and $P' = R_m(P)$ and $P'' = R_O(P')$. Then $OP = OP' = OP''$ (the first equality because m is the perpendicular bisector of PP' by definition of the reflection and because $O \in m$, and the second by definition of the rotation about O). Thus $\Delta POP''$ is isosceles of base PP'' and therefore the perpendicular bisector of PP'' passes through the top vertex O, and gives us q for the reflection R_q such that $R_q(P) = P''$.

Theorem *Theorem:*

The composition of two rotations with different centers and different angles is either another rotation or a translation.

We can use Property 3 and the special line m which passes through the two centers, once as the first reflection (in the rotation decomposition) and then as the second reflection. $R_{O_1} = R_m \circ R_p$ (where p is determined from R_{O_1} itself and from m), and we also write $R_{O_2} = R_q \circ R_m$ (with q again determined from R_{O_2} itself and from m, all these being given). When we compose the rotations, we can use Property 2 to obtain just two reflections, and again either Property 3 to get back to the rotation, or Property 4 to obtain the translation. That is: $R_{O_2} \circ R_{O_1} = R_q \circ R_m \circ R_m \circ R_p = R_q \circ R_p$.

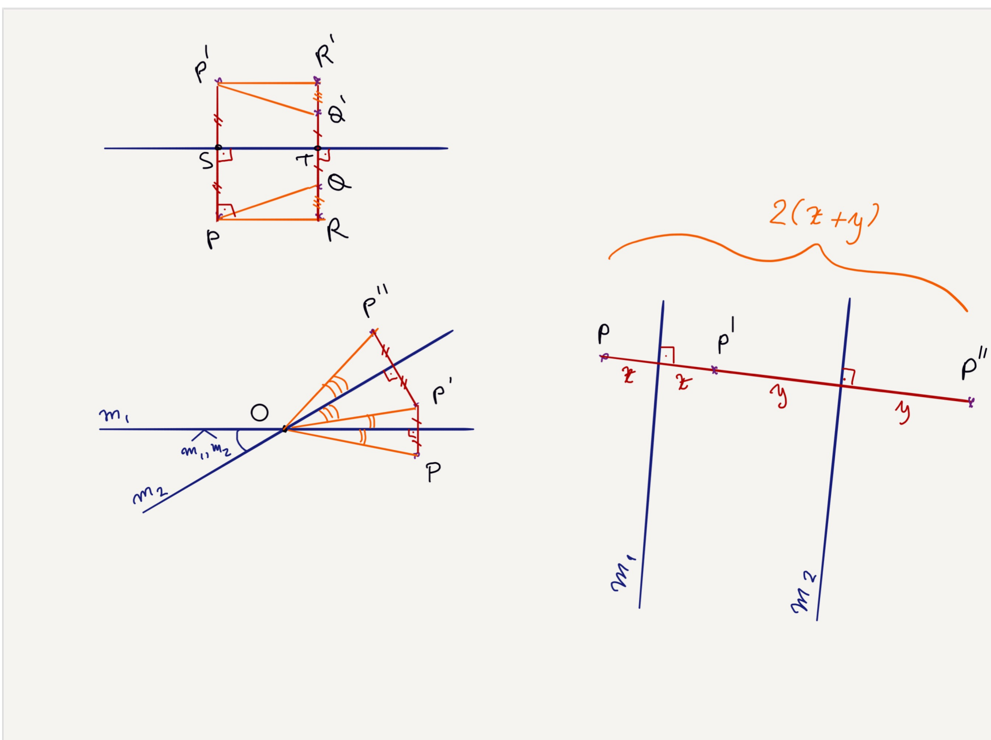


Figure 10.3: Reflections

10.2 Homework

1. *$\triangle ABC$ with $A(-2,2)$, $B(-2,5)$, $C(-5,1)$. is reflected over the line $x=1$. Graph the image.*
2. *(Constructing a Rotation) Use a straightedge, compass, and protractor to rotate $\triangle ABC$ 60 clockwise about point $O(0,0)$*

   Hint: Place the point of the compass at O and draw an arc clockwise from point A. Use the protractor to measure a 60 angle $\widehat{AOA'}$ Label the point A'.
3. *Draw on a coordinate system the $\triangle ABC$ with $A(2,2)$, $B(8,8)$, $C(6,10)$.*
   - (a) *Reflect the given triangle, i.e. your pre-image, over $y=-1$ followed by $y=-7$. Write the new coordinates.*
   - (b) *Which is the transformation equivalent to this double reflection?*
   - (c) *Which translation would move the new image back to the pre-image?*
   - (d) *Start over. Reflect the pre-image over $y=-7$, then $y=-1$. How is this different from (a)?*
   - (e) *Write the rules for (a) and (d). How do they differ?*
4. *Let m be the line $x=1$. Find where the reflection R_m would send points $(2,3)$, $(5,0)$.*
5. *Same questions for reflection R_p with p the line $x=3$.*
6. *Where would the composition $R_p \circ R_m$ (that is, first reflect about m and then about p) send the points $(2,3)$, $(5,0)$?*
7. *Write the general formula for $R_m(x,y)$ (hint: write $x=1+a$ for some a). Same question for R_p.*
8. *Draw on a coordinate system the square $ABCD$ with $A(6,-2)$, $B(10,-4)$, $C(8,-8)$ and $D(4,-6)$*
   - (a) *Reflect the square over $y=x$, followed by a reflection over the $x$axis.*
   - (b) *Which is the transformation equivalent to this double reflection ?*
9. *Can you find an example showing that the composition of two isometries is not commutative ? Think about "flipping" your smartphone, and also about rotating it.*
10. *Draw on a coordinate system the $\triangle ABC$ with $A(8,10)$, $B(2,6)$, and $C(10,4)$. Reflect the triangle over $y=3$ and $y=-5$.*
11. *Review questions:*
    - (a) *What transformation is equivalent to a reflection over two parallel lines?*
    - (b) *What transformation is equivalent to a reflection over two perpendicular lines?*
    - (c) *What transformation is equivalent to a reflection over two intersecting lines?*