Classwork 3 + Homework 3
Combination, Pascal's Triangle

Math 7a

October 6, 2017

						n=0		1						
					n=1		1		1					
				n=2		1		2		1				
			n=3		1		3		3		1			
		n=4		1		4		6		4		1		
	n=5		1		5		10		10		5		1	
n=6		1		6		15		20		15		6		1

Let's motivate our lesson by asking a simple question: how many distinct teams of 2 students can our class of 10 students send to a tournament? This is equivalent to the number of distinct combinations of size 2 that we can make from 10 distinct items. (*hint:* $10 * 9$ *would be incorrect!*)

Now let's imagine that we had to send a team of 4 students out of 10 in class. Following our intuitive reasoning of selecting 1st team member in 10 ways, 2nd team member in 9 ways, and so on we arrive at $10 * 9 * 8 * 7$. But teams composed of students $\{1, 2, 3, \text{ and } 4\}$ and $\{2, 1, 3, \text{ and } 4\}$ are really the same teams and we've double-counted them! And there are many more of them: in fact we can reorder students 1, 2, 3, and 4 in $4 * 3 * 2 * 1 = 4!$ ways and still end up with the same team. That means our naive guess needs to be corrected by dividing by 4! giving us:

$$\text{number of possible teams} = \frac{10 * 9 * 8 * 7}{4 * 3 * 2 * 1} = \frac{(10 * 9 * 8 * 7) * (6 * 5 * 4 * 3 * 2 * 1)}{(4 * 3 * 2 * 1) * (6 * 5 * 4 * 3 * 2 * 1)} = \frac{10!}{4!6!} = 210$$

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We can obtain the same result from our Pascal triangle by looking up row n and column k. Pascal's triangle is constructed by starting at the top with a single entry of 1 and at each new row adding entries from previous row immediately to the left and right. When there is a blank space in the row above, we treat it as 0.

Considering the general case of class of n students and team of k students, we get a convenient formula: $\frac{n!}{k!(n-k)!}$. We can even verify the formula using our Pascal Triangle:

$$C_{5,2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10$$

Just as in the triangle!

We will often use the notation $C_{n,k}$ from above for indicating the number of combinations of k items possible from set of n items. This is also frequently noted as $\binom{n}{k}$ and termed "n-choose-k" (from n choose k), where $n \geq k$ is true. (*No way of choosing more than n items from set of n items!*)

It is important to note that while reading the triangle, counting for both n (number of items to choose from) and k (number of items chosen) start at 0:

- There is $C_{n,0} = \binom{n}{0} = 1$ way of selecting 0 items from any number of items: the empty set ($\varnothing$).
- There is $C_{0,0} = \binom{0}{0} = 1$ way of selecting anything from 0 items: the empty set ($\varnothing$).
- There is also exactly $C_{n,n} = \binom{n}{n} = 1$ way of selecting all n items from a set of n.

You may have noticed that each row of the triangle can be quickly obtained from the row above by adding each pair of consecutive entries:

$$C_{n,k} = C_{n-1,k-1} + C_{n-1,k}$$

Let's try expanding $(a+b)^2$. We get our familiar $a^2 + 2ab + b^2$. Note that the coefficients $\{1, 2, 1\}$ resemble the 2nd row of Pascal triangle exactly. But how about expanding $(a+b)^5$? This amounts to finding the coefficient in front of a^5, a^4b, a^3b^2 and so on. Looking up the row $n = 5$ of our Pascal triangle, we see a sequence of numbers: 1, 5, 10, 10, 5, 1. This suggests:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

And it happens to be true. Moreover, nth row of the Pascal triangle happens to be the coefficients of $(a+b)^n$.

1. From a class of 3 girls and 7 boys, how many distinct teams of 2 girls and 2 boys can we make? Remember that for each configuration of the girls' sub-team, there can be many configurations of the boys' sub-team.

2. Find the number of distinct paths a King can take going from bottom-left corner to top-right corner of an 8x8 chessboard if it is only allowed to move up (U) or right (R). (*hint: no matter what the exact path is, the King has to move up exactly 7 times*)

3. Is there a pattern for the sum of all entries of nth row of Pascal triangle? In other words, is there a general formula for $S_n = C_{n,0} + C_{n,1} + ... + C_{n,n}$?

4. You are packing for a week long summer science camp and need to bring exactly 7 pairs of socks and 7 t-shirts.
    (a) If you happen to own 9 pairs of socks and 10 t-shirts, how many distinct suitcases can you pack?
    (b) How many distinct suitcases can you pack if you are allowed to overpack and bring no less than 7 pairs of socks and 7 t-shirts?

5. Math A meets in room 101 and Math B meets in room 102. We know that both Math A and B have only 4 students each and room 101 has one extra chair compared to room 102. If the number of possible combinations of unoccupied chairs in room 101 during Math A is twice as large as number of possible combinations of unoccupied chairs in room 102 during Math B, find the number of chairs in each room.

6. Is $C_{n,k}$ always equal to $C_{n,n-k}$? If you think so, explain why it would be true no matter what the values of n and k are. Otherwise, provide n and k such that the equality is violated: $C_{n,k} \neq C_{n,n-k}$.

7. Imagine a non-conventional Olympic tournament where every player has to play once with every other player instead of the conventional single-elimination.

   (a) If 16 athletes entered the competition, how many matches will take place?

   (b) Now the tournament is modified to include multiple stages. After each stage of the tournament, the bottom half of all players leave the competition and the top half continue to the next stage. If the same 16 players enter the tournament and at each stage every player still has to compete once against every other player left in tournament, how many matches in total need to be played until a single athlete is left and is declared the champion?